

ILL-POSEDNESS OF THE CUBIC NONLINEAR HALF-WAVE EQUATION AND OTHER FRACTIONAL NLS ON THE REAL LINE

ANTOINE CHOFFRUT AND OANA POCOVNICU

ABSTRACT. In this paper, we study ill-posedness of cubic fractional nonlinear Schrödinger equations. First, we consider the cubic nonlinear half-wave equation (NHW) on \mathbb{R} . In particular, we prove the following ill-posedness results: (i) failure of local uniform continuity of the solution map in $H^s(\mathbb{R})$ for $s \in (0, \frac{1}{2})$, and also for $s = 0$ in the focusing case; (ii) failure of C^3 -smoothness of the solution map in $L^2(\mathbb{R})$; (iii) norm inflation and, in particular, failure of continuity of the solution map in $H^s(\mathbb{R})$, $s < 0$. By a similar argument, we also prove norm inflation in negative Sobolev spaces for the cubic fractional NLS. Surprisingly, we obtain norm inflation *above* the scaling critical regularity in the case of dispersion $|D|^\beta$ with $\beta > 2$.

1. INTRODUCTION

In this paper, we consider the fractional nonlinear Schrödinger equations (NLS) on \mathbb{R} :

$$\begin{cases} i\partial_t u - |D|^\beta u = \mu |u|^2 u & (t, x) \in \mathbb{R} \times \mathbb{R}, \\ u(0) = u_0, \end{cases} \quad (1.1)$$

where u is complex-valued, $\beta > 0$, $\widehat{|D|^\beta f}(\xi) = |\xi|^\beta \widehat{f}(\xi)$ for all $\xi \in \mathbb{R}$, and $\mu \in \{-1, 1\}$. When $\mu = 1$ the equation (1.2) is called defocusing, while it corresponds to the focusing equation when $\mu = -1$. PDEs with a nonlocal dispersion, such as (1.1), appear in various physical contexts. For example, such one-dimensional PDEs were proposed by Majda, McLaughlin, and Tabak as models for wave turbulence in [42]. Other physical instances include water waves [31], continuum limits of lattice points [35], and gravitational collapse [19, 23].

A major part of this paper is dedicated to the study of ill-posedness of the following cubic nonlinear half-wave equation (NHW) on \mathbb{R} , corresponding to the case $\beta = 1$ in (1.1):

$$\begin{cases} i\partial_t u - |D|u = \mu |u|^2 u \\ u(0) = u_0. \end{cases} \quad (1.2)$$

In particular, we prove different forms of ill-posedness depending on the phase space $H^s(\mathbb{R})$; see Theorem 1.3. In the last section, we return to the cubic fractional NLS (1.1) and prove a strong form of ill-posedness in negative Sobolev spaces; see Theorem 1.5.

We start by recalling briefly the notions of local and global well-posedness for (1.1).

Definition 1.1. We say that the Cauchy problem (1.1) is locally well-posed in $H^s(\mathbb{R})$ if the following two conditions hold:

2010 *Mathematics Subject Classification.* 35L05, 35L60.

Key words and phrases. nonlinear wave equation; nonlinear fractional Schrödinger equation; ill-posedness; norm inflation.

(i) For every bounded set B of $H^s(\mathbb{R})$, there exist $T > 0$ and a Banach space X_T continuously embedded in $C([-T, T]; H^s(\mathbb{R}))$ such that if $u_0 \in B$, then there exists a unique solution u of (1.1) on $[-T, T]$ in X_T .

(ii) The solution map $\Phi_\mu : u_0 \mapsto u$ is continuous from B to $C([-T, T]; H^s(\mathbb{R}))$.

If T can be chosen arbitrarily large, then we say that the Cauchy problem (1.1) is globally well-posed in $H^s(\mathbb{R})$.

1.1. The cubic nonlinear half-wave equation. NHW arises as a Hamiltonian evolution associated to the energy:

$$E(u(t)) := \int \frac{1}{2} |D|^{\frac{1}{2}} u(t, x)|^2 + \frac{\mu}{4} |u(t, x)|^4 dx.$$

In particular, the energy is conserved (at least formally) by the evolution: $E(u(t)) = E(u(0))$ for all $t \in \mathbb{R}$. Notice that the energy $E(u)$ is positive-definite when $\mu = 1$ and, moreover, it controls the $\dot{H}^{\frac{1}{2}}(\mathbb{R})$ -norm of u , while it is not sign definite when $\mu = -1$. Not surprisingly, as we see in Proposition 1.2 below, μ plays a crucial role in the global-in-time dynamics of (1.2). Another important conservation law of (1.2) is the mass:

$$M(u(t)) := \int_{\mathbb{R}} |u(t, x)|^2 dx = M(u(0)).$$

The symmetries associated to the two conservation laws above, as guaranteed by Noether's theorem, are the time translation $u(t, x) \mapsto u(t + t_0, x)$ and the phase rotation $u(t, x) \mapsto e^{i\theta} u(t, x)$. Another crucial symmetry of (1.2) is the scaling invariance. Namely, if u is a solution of (1.2), then

$$u_\lambda(t, x) = \lambda^{\frac{1}{2}} u(\lambda t, \lambda x)$$

is also a solution of (1.2) with rescaled initial data $u_\lambda(0, x) = \lambda^{\frac{1}{2}} u(0, \lambda x)$. Notice that the scaling leaves the L^2 -norm of a solution invariant, $\|u_\lambda(t, \cdot)\|_{L^2} = \|u(\lambda t, \cdot)\|_{L^2}$. Namely, (1.2) is said to be “ L^2 -critical” or “mass-critical”.

We first recall the well-posedness theory for (1.2) from [26, 38, 49].

Proposition 1.2 ([26, 38, 49], Well-posedness theory for NHW). Let $s \geq \frac{1}{2}$.

The Cauchy problem (1.2) is locally well-posed in $H^s(\mathbb{R})$. Moreover, for $s > \frac{1}{2}$, the solution map Φ_μ is Lipschitz continuous on bounded subsets of $H^s(\mathbb{R})$.

In the defocusing case ($\mu = 1$), (1.2) is globally well-posed in $H^s(\mathbb{R})$. In the focusing case ($\mu = -1$), (1.2) is globally well-posed for initial data in $H^s(\mathbb{R})$ of sufficiently small mass, while there exist finite time blowup solutions of large mass.

1.2. Ill-posedness of the cubic nonlinear half-wave equation. One of our goals in the present paper is to complete the local-in-time study of the Cauchy problem for NHW on \mathbb{R} . In particular, we complement Proposition 1.2 by proving various ill-posedness results in $H^s(\mathbb{R})$ for $s < \frac{1}{2}$.

Given $s \in \mathbb{R}$, in the following we denote by $\Phi_\mu : u_0 \mapsto u$ the a priori defined solution map of (1.2) from $H^s \cap \mathcal{S}(\mathbb{R})$ to $C([-T, T]; H^s(\mathbb{R}) \cap \mathcal{S}(\mathbb{R}))$ (see Definition 1.1 above). (Here, $\mathcal{S}(\mathbb{R})$ denotes the space of Schwartz functions on \mathbb{R} .) Given $t \in \mathbb{R}$, we also denote by $\Phi_\mu(t) : u_0 \mapsto u(t)$ the solution map acting on $H^s(\mathbb{R}) \cap \mathcal{S}(\mathbb{R})$.

Theorem 1.3 (Ill-posedness of NHW on \mathbb{R}).

(i) (*Failure of local uniform continuity, $0 < s < \frac{1}{2}$*). Let $0 < s < \frac{1}{2}$. Φ_μ fails to be uniformly continuous on bounded sets of $H^s(\mathbb{R})$, for $\mu \in \{-1, 1\}$. More precisely, given $0 < \varepsilon \ll 1$, there exist global solutions of (1.2) u_1^ε and u_2^ε such that $\|u_1^\varepsilon(0)\|_{H^s} \lesssim 1$, $\|u_2^\varepsilon(0)\|_{H^s} \lesssim 1$,

$$\lim_{\varepsilon \rightarrow 0} \|u_1^\varepsilon(0) - u_2^\varepsilon(0)\|_{H^s} = 0,$$

and

$$\liminf_{\varepsilon \rightarrow 0} \|u_1^\varepsilon - u_2^\varepsilon\|_{L^\infty([0,T];H^s)} \gtrsim 1 \quad \text{for all } T > 0.$$

(ii) (*Focusing NHW, $s = 0$*). Φ_{-1} fails to be uniformly continuous on bounded sets of $L^2(\mathbb{R})$. More precisely, a statement similar to that in (i) holds with $\mu = -1$ and $s = 0$.

(iii) (*Failure of C^3 -smoothness, $s = 0$*). Fix $0 < t \leq 1$. $\Phi_\mu(t)$ fails to be C^3 -smooth on $L^2(\mathbb{R})$ for $\mu \in \{-1, 1\}$.

(iv) (*Norm inflation in H^s , $s < 0$*). Let $s < 0$. Given $0 < \varepsilon \ll 1$, there exist a solution $u^\varepsilon \in C(\mathbb{R}; H^\infty(\mathbb{R}))$ and $0 < t_\varepsilon < \varepsilon$ such that

$$\|u^\varepsilon(0)\|_{H^s(\mathbb{R})} < \varepsilon \quad \text{and} \quad \|u^\varepsilon(t_\varepsilon)\|_{H^s(\mathbb{R})} > \frac{1}{\varepsilon}.$$

In particular, Φ_μ fails to be continuous at zero in $H^s(\mathbb{R})$, $s < 0$, for $\mu \in \{-1, 1\}$.

In the periodic setting, Georgiev, Tzvetkov, and Visciglia [24] recently proved the failure of local uniform continuity of the solution map of NHW in $H^s(\mathbb{T})$, $s \in (\frac{1}{4}, \frac{1}{2})$. Our proof of Theorem 1.3 (i) is inspired by their work. There are, however, some differences discussed in Subsection 1.5 below, allowing us to cover a larger range of regularities $s \in (0, \frac{1}{2})$.

In view of Definition 1.1, we note that parts (i)-(iii) of Theorem 1.3 refer to a ‘mild’ form of ill-posedness, namely the failure of local uniform continuity or C^3 -smoothness of the solution map. In particular, it might still be possible to construct a locally continuous flow in $H^s(\mathbb{R})$ for $s \in [0, \frac{1}{2})$. Parts (i)-(iii), on the other hand, show that such a local well-posedness result in $H^s(\mathbb{R})$, $s \in [0, \frac{1}{2})$, cannot be proved using a fixed point argument.

Theorem 1.3 (iii) states that the solution map of the defocusing NHW fails to be C^3 -smooth on $L^2(\mathbb{R})$. We do not know, however, whether the solution map fails to be uniformly continuous on bounded sets of $L^2(\mathbb{R})$. This remains an interesting open question. See Remark 3.4 for a detailed discussion.

As remarked in the statement of Theorem 1.3, the norm inflation in part (iv) is a stronger property than the failure of continuity of the solution map Φ_μ at zero in $H^s(\mathbb{R})$. Indeed, for the failure of continuity of Φ_μ at zero in $H^s(\mathbb{R})$ it suffices to find a solution $u^\varepsilon \in C(\mathbb{R}, H^\infty(\mathbb{R}))$ and $0 < t_\varepsilon < \varepsilon$ such that $\|u^\varepsilon(0)\|_{H^s} < \varepsilon$ and $\|u^\varepsilon(t_\varepsilon)\|_{H^s} \gtrsim 1$. (Here, $H^\infty(\mathbb{R}) = \cap_{s>0} H^s(\mathbb{R})$.)

1.3. Relation to the cubic Szegő equation. An important feature of the nonlinear half-wave equation (1.2) is the fact that the resonant equation associated to it is a completely integrable model, namely the cubic Szegő equation:

$$\begin{cases} i\partial_t V = \Pi_+(|V|^2 V) \\ V(0) = V_0, \end{cases} \quad (1.3)$$

where $\widehat{\Pi_+ f}(\xi) = \widehat{f}(\xi)\mathbf{1}_{\xi \geq 0}$ is the Szegő projector. In particular, the dynamics of NHW can be well approximated, for a long time, by that of the Szegő equation. See [49] and Proposition 2.6 below. This approximation plays an important role in the proof of Theorem 1.3 (i).

We recall that the cubic Szegő equation was introduced by Gérard and Grellier in [25] on \mathbb{T} , and was studied on \mathbb{R} by the second author in [47, 48]. The Szegő equation on \mathbb{R} is globally well-posed in

$$H_+^s(\mathbb{R}) := L_+^2(\mathbb{R}) \cap H^s(\mathbb{R}), \quad s \geq \frac{1}{2},$$

where $L_+^2(\mathbb{R}) = \{f \in L^2(\mathbb{R}) : \text{supp } \widehat{f} \subset [0, \infty)\}$. We remark that, via a Paley-Wiener theorem, $L_+^2(\mathbb{R})$ can also be identified with the Hardy space of holomorphic functions in the upper-half plane $\mathbb{C}_+ := \{z : \text{Im } z > 0\}$.

As a byproduct of the proof of Theorem 1.3, we also have the following ill-posedness result for the cubic Szegő equation (1.3).

Proposition 1.4 (Ill-posedness of the cubic Szegő equation on \mathbb{R}).

- (i) Let $0 \leq s < \frac{1}{2}$. The solution map of the cubic Szegő equation fails to be uniformly continuous on bounded sets in $H_+^s(\mathbb{R})$.
- (ii) Let $s < 0$. The cubic Szegő equation (1.3) has the norm inflation property in $H_+^s(\mathbb{R})$ (in the sense of part (iv) of Theorem 1.3). In particular, the solution map of the cubic Szegő equation (1.3) fails to be continuous at zero in $H_+^s(\mathbb{R})$, $s < 0$.

Along with the above mentioned approximation of NHW by the Szegő equation, Proposition 1.4 (i) (for $0 < s < \frac{1}{2}$) is the key ingredient in proving Theorem 1.3 (i). On the other hand, Proposition 1.4 (ii) follows by slightly modifying the argument used to prove Theorem 1.3 (iv).

1.4. Norm inflation in negative Sobolev spaces for fractional NLS. Now we turn our attention to the cubic fractional NLS (1.1) with a general value of $\beta > 0$. The equation (1.1) possesses the scaling symmetry $u \mapsto u_\lambda(t, x) := \lambda^{\frac{\beta}{2}} u(\lambda x, \lambda^\beta t)$. Associated to this symmetry, one defines the scaling critical regularity s_{crit} to be the index s for which the \dot{H}^s -norm of $u(0)$ is invariant under this scaling. A simple calculation yields that

$$s_{\text{crit}} = \frac{1 - \beta}{2}.$$

The well-posedness theory of the cubic fractional NLS (1.1) with $1 < \beta < 2$ was studied in [14, 18]. In particular, it was shown that (1.1) is locally well-posed in H^s for $s \geq \frac{2-\beta}{4}$ both on \mathbb{R} and \mathbb{T} . Moreover, [14] proved the failure of local uniform continuity of the solution map of (1.1) on \mathbb{R} in $H^s(\mathbb{R})$, $\frac{2-3\beta}{4(\beta+1)} < s < \frac{2-\beta}{4}$, in the case $1 < \beta < 2$.

Proceeding as in Theorem 1.3 (iv), we prove the following norm inflation property (a strong form of ill-posedness) for fractional NLS with a general $\beta > 0$.

Theorem 1.5 (Norm inflation property for fractional NLS). *The cubic fractional NLS (1.1) has the norm inflation property in $H^s(\mathbb{R})$ (in the sense of Theorem 1.3 (iv)) in the following cases (see the shaded region in Figure 1):*

- $0 < \beta < 1$: $s < 0$,
- $1 \leq \beta < 2$: $s < s_{\text{crit}}$,
- $\beta = 2$: $s \leq s_{\text{crit}}$,
- $\beta > 2$: $s < \frac{1-2\beta}{6}$.

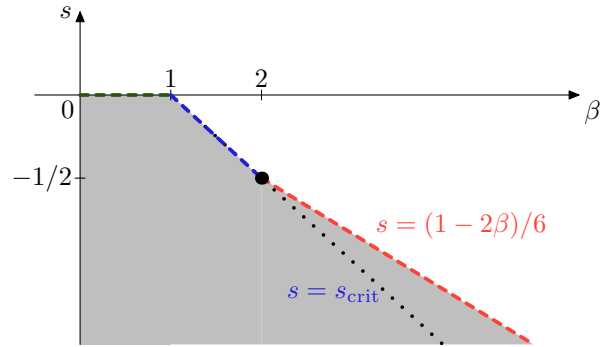


FIGURE 1. Region of norm inflation for fractional NLS.

It is generally conjectured that a PDE is ill-posed in H^s for $s < s_{\text{crit}}$. For cubic NLS ($\beta = 2$) on \mathbb{R} and \mathbb{T} , Christ, Colliander, and Tao [16] and Kishimoto [36] proved, indeed, norm inflation in $H^s(\mathbb{R})$, $s \leq s_{\text{crit}} = -\frac{1}{2}$. In Theorem 1.5, we prove norm inflation in $H^s(\mathbb{R})$, $s < s_{\text{crit}}$, for all fractional NLS with $1 \leq \beta \leq 2$. Surprisingly, in the case $\beta > 2$, we obtain norm inflation for regularities $s < \frac{1-2\beta}{6}$, where we note that $\frac{1-2\beta}{6} > s_{\text{crit}}$. To the best of the authors' knowledge, this is the first result of norm inflation *above* the scaling critical regularity for NLS-type equations with a gauge-invariant nonlinearity.

In the non gauge-invariant case, previously Iwabuchi and Ogawa [32] and Iwabuchi and Uriya [33] obtained norm inflation above the scaling critical regularity for NLS with nonlinearities u^2 on \mathbb{R} , and $|u|^2$ on \mathbb{R}^n , $n = 1, 2, 3$, respectively. Our norm inflation results are inspired by [32] and by a further development by Kishimoto [36].

In the gauge-invariant case, there are results on other types of ill-posedness above the scaling critical regularity. We explain below why Theorem 1.5 with $\beta > 2$ is of a different nature compared to these previous results. Previously, Christ, Colliander, and Tao [17] and Molinet [43] showed the failure of continuity of the solution map in $H^s(\mathbb{T})$, $s < 0$, for the cubic NLS on \mathbb{T} ($\beta = 2$ and $s_{\text{crit}} = -\frac{1}{2}$). Furthermore, Guo and Oh [28] proved non-existence of solutions of the cubic NLS on \mathbb{T} if the initial data is in $H^s(\mathbb{T}) \setminus L^2(\mathbb{T})$, where $s \in (-\frac{1}{8}, 0)$. The latter also holds for the cubic biharmonic NLS ($\beta = 4$) on \mathbb{T} (with $s_{\text{crit}} = -\frac{3}{2}$). We remark that all these ill-posedness results no longer apply if we remove a

certain resonant term from the nonlinearity. More precisely, set $\mathcal{M}(u) := \frac{1}{2\pi} \int_{\mathbb{T}} |u|^2 dx$ and consider the Wick ordered cubic fractional NLS on \mathbb{T} :

$$i\partial_t u - |D|^\beta u = (|u|^2 - \mathcal{M}(u)) u.$$

Then the above-mentioned ill-posedness results in [17, 43, 28] do not apply to the Wick ordered cubic NLS, nor to the Wick ordered cubic biharmonic NLS.

Let us now turn to Theorem 1.5. First, we make the observation that the proof of Theorem 1.5 for (1.1) on \mathbb{R} can be carried over to \mathbb{T} almost literally. One can then use the gauge transformation to map a solution u of the cubic fractional NLS (1.1) on \mathbb{T} with initial data $u_0 \in L^2(\mathbb{T})$, into a solution of the Wick ordered cubic fractional NLS on \mathbb{T} :

$$u \mapsto \mathcal{G}(u)(t) := e^{-2it\mathcal{M}(u_0)} u(t).$$

Note that \mathcal{G} preserves any H^s -norm. Since we only work with smooth initial data, this immediately allows us to deduce that the norm inflation in Theorem 1.5 also holds for the Wick ordered cubic fractional NLS on \mathbb{T} . In particular, this shows that simply removing $\mathcal{M}(u_0)u$ from the nonlinearity of the cubic fractional NLS on \mathbb{T} is not sufficient to make the solution map continuous. Instead, in our case, the discontinuity of the solution map is more critical and is due to a genuinely nonlinear effect (the “high-to-low frequency cascade” in the nonlinearity; see Subsection 1.5 below). To the best of the authors’ knowledge, this is a new phenomenon for regularities larger than the scaling critical regularity, in the case of gauge-invariant nonlinearities.

1.5. Comments on the proofs and further remarks.

In a recent work, Georgiev, Tzvetkov, and Visciglia [24] proved the failure of local uniform continuity of the solution map of NHW (1.2) on the torus \mathbb{T} in $H^s(\mathbb{T})$, $s \in (\frac{1}{4}, \frac{1}{2})$. In proving Theorem 1.3 (i), we follow the strategy of [24]. Namely, given $0 < \varepsilon \ll 1$, we consider two solutions $u_1^\varepsilon, u_2^\varepsilon$ of NHW that can be respectively approximated by explicit traveling waves $v_1^\varepsilon, v_2^\varepsilon$ of the Szegő equation (1.3). The two traveling waves $v_1^\varepsilon, v_2^\varepsilon$ are chosen in such a way that they are very close to each other at time $t = 0$, but after a long time their distance increases *relatively* to their initial distance (remaining still small). Then, the solutions $u_1^\varepsilon, u_2^\varepsilon$ of NHW inherit the same properties regarding their distance. Finally, exploiting the scaling symmetry of NHW on \mathbb{R} , the distance between the solutions $u_1^\varepsilon, u_2^\varepsilon$ of NHW can be made $\gtrsim 1$ at a very small time. While the result in [24] in the periodic case holds in $H^s(\mathbb{T})$, $s \in (\frac{1}{4}, \frac{1}{2})$, we were able to prove failure of the local uniform continuity of the solution map of NHW on \mathbb{R} in $H^s(\mathbb{R})$ for $s \in (0, \frac{1}{2})$. We enlarged the range of regularities by using a finer approximation of NHW by the Szegő equation, together with the scaling symmetry of NHW on \mathbb{R} (not available on \mathbb{T}). See Remark 2.8 below for details.

For the proof of Theorem 1.3 (ii), we work directly with two traveling waves of the focusing NHW on \mathbb{R} , rather than working with solutions that can be approximated by explicit solutions of the Szegő equation. Note that this argument is not applicable to the defocusing NHW, since there are no traveling waves in that case. We choose the two speeds of the traveling waves of focusing NHW in such a way that: (i) at time $t = 0$ the traveling waves are very close to each other, and (ii) at a later time they are spatially located far

away from each other and, therefore, their distance becomes $\gtrsim 1$. This type of argument was first used in the work of Birnir, Kenig, Ponce, Svanstedt, and Vega [5]. It was later used by Kenig, Ponce, and Vega [34] to prove the failure of local uniform continuity of the solution map for the focusing cubic NLS on \mathbb{R} . See also [4, 29].

The failure of C^3 -smoothness of the solution map in Theorem 1.3 (iii) follows from the unboundedness of the trilinear operator $\nabla^3 \Phi_\mu|_{u_0=0}$. This type of approach was first introduced by Bourgain in [7], where he proves the failure of C^3 -smoothness of the solution map for KdV and mKdV on both \mathbb{T} and \mathbb{R} below certain threshold regularity. See also [50].

The proof of Theorem 1.3 (iv) is based on analyzing each term appearing in the Picard iteration scheme for NHW (1.2). In particular, we show that the cubic term U_3 (see equation (5.3)) dominates all the other terms and is unbounded in $H^s(\mathbb{R})$. To achieve this goal we exploit the “high-to-low energy cascade” in the nonlinearity. This idea appeared first in the work of Bejenaru and Tao [2] as an abstract and general argument for proving ill-posedness. Recently, Iwabuchi and Ogawa [32] developed this idea further, making it more easily applicable to a wider class of equations. In this paper, we follow closely an argument of Kishimoto [36]. We also exploit the abundance of resonances of NHW. In Theorem 1.5, we prove an analogous result for the cubic fractional NLS with general $\beta > 0$. More care is needed in this case since (1.1) with $\beta \neq 1$ has less resonances than NHW.

Remark 1.6. In [46], Oh further extended the approach of Iwabuchi and Ogawa [32] to prove a norm inflation phenomenon based at general initial conditions (not only the zero initial condition) for the cubic NLS on \mathbb{R}^d and \mathbb{T}^d . It was remarked in [46] that a similar argument can be used to extend our results from Theorem 1.3 (iv) and Theorem 1.5 to norm inflation based at general initial conditions. See also [52] for a previous result on generic ill-posedness.

Remark 1.7. Theorem 1.3 (iv) and Theorem 1.5 also hold on \mathbb{T} with essentially the same proof. In a recent work, Oh and Wang [45] obtained a similar result for the fractional NLS on \mathbb{T} . On the one hand, their result does not include regularities above s_{crit} . On the other hand, it includes positive regularities $0 < s < s_{\text{crit}}$ (for $0 < \beta < 1$), that are not considered in this paper. The strategy of [45] is different from ours. It consists in exploiting the “high-to-low energy cascade” in the dynamics of dispersionless NLS $i\partial_t u = |u|^2 u$, and then in approximating dispersionless NLS by the small dispersion fractional NLS $i\partial_t u - \nu|D|^\beta u = |u|^2 u$, $0 < \nu \ll 1$. This strategy was first used by Christ, Colliander, and Tao [15, 16] in the context of NLS on \mathbb{R}^d . See also [9, 10]. In [45], this was adapted to the periodic setting by using a modified scaling argument to relate small dispersion fractional NLS to fractional NLS. See also [36, 13] for recent results on norm inflation for NLS in the periodic setting.

Finally, we refer to [8, 37, 51, 41, 40, 39, 30, 1, 11, 12] for more ill-posedness results for nonlinear PDEs.

Organization of the paper. Sections 2-5 are dedicated to the proof of Theorem 1.3. More precisely, the four parts of Theorem 1.3 follow from Propositions 2.7, 3.1, 4.1, and 5.7, respectively. We also note that Proposition 1.4 on ill-posedness of the cubic Szegő

equation follows from Corollary 2.4 and Remarks 4.2 and 5.10. Finally, in Section 6 we prove Theorem 1.5.

2. FAILURE OF LOCAL UNIFORM CONTINUITY OF THE SOLUTION MAP OF NHW IN $H^s(\mathbb{R})$, $s \in (0, \frac{1}{2})$

In this section, we prove Theorem 1.3 (i), namely the failure of local uniform continuity of the solution map of (1.2) with $\mu = \pm 1$, with respect to the topology of $H^s(\mathbb{R})$, $s \in (0, \frac{1}{2})$.

As it was recalled in the introduction, solutions of equation (1.2) with small, well-prepared data can be approximated for a long time by solutions of the completely integrable cubic Szegő equation (1.3). See Proposition 2.6 below. Thus, we will use explicit traveling waves of this integrable model to first show the failure of local uniform continuity for the Szegő equation. Next, we will use the above mentioned approximation to deduce the failure of local uniform continuity for (1.2).

We start with an elementary lemma that we use repeatedly in this paper. We denote by \mathcal{F} the Fourier transform:

$$\mathcal{F}f(\xi) = \hat{f}(\xi) := \int_{\mathbb{R}} e^{-ix\xi} f(x) dx \quad \text{for all } \xi \in \mathbb{R}.$$

For $s \in \mathbb{R}$, we define the homogeneous \dot{H}^s -norm by

$$\|f\|_{\dot{H}^s(\mathbb{R})} := \frac{1}{\sqrt{2\pi}} \left(\int_{\mathbb{R}} |\xi|^{2s} |\hat{f}(\xi)|^2 d\xi \right)^{\frac{1}{2}},$$

while the non homogeneous H^s -norm is defined by

$$\|f\|_{H^s(\mathbb{R})} := \frac{1}{\sqrt{2\pi}} \left(\int_{\mathbb{R}} \langle \xi \rangle^{2s} |\hat{f}(\xi)|^2 d\xi \right)^{\frac{1}{2}},$$

where $\langle \xi \rangle = \sqrt{1 + \xi^2}$.

Lemma 2.1. *Let $c \in \mathbb{R}$ and $p > 0$. Then the following hold:*

- (i). $\mathcal{F} \left(\frac{1}{x-ct+ip} \right) (\xi) = -2\pi i e^{-ict\xi} e^{-p\xi} \mathbf{1}_{\xi > 0}$ for all $\xi \in \mathbb{R}$.
- (ii). For any $s > -\frac{1}{2}$, we have $\left\| \frac{1}{x-ct+ip} \right\|_{\dot{H}^s} = \frac{\sqrt{2\pi\Gamma(2s+1)}}{(2p)^{s+\frac{1}{2}}}$, where Γ denotes the gamma function.
- (iii). $\frac{1}{x-ct+ip} \notin H^s(\mathbb{R})$ for any $s \leq -\frac{1}{2}$.

Proof. Part (i) follows directly from the residue theorem, while part (ii) follows from (i) and by the change of variables $\eta = 2p\xi$:

$$\begin{aligned} \left\| \frac{1}{x-ct+ip} \right\|_{\dot{H}^s}^2 &= \frac{1}{2\pi} \int |\xi|^{2s} \left| \mathcal{F} \left(\frac{1}{x-ct+ip} \right) (\xi) \right|^2 d\xi = 2\pi \int_0^\infty \xi^{2s} e^{-2p\xi} d\xi \\ &= \frac{2\pi}{(2p)^{2s+1}} \int_0^\infty \eta^{2s} e^{-\eta} d\eta = \frac{2\pi\Gamma(2s+1)}{(2p)^{2s+1}}. \end{aligned}$$

For part (iii), it suffices to notice that $\int_0^\infty \eta^{2s} e^{-\eta} d\eta = \infty$ for any $s \leq -\frac{1}{2}$. □

Next we recall the following classification of traveling waves of the Szegő equation on \mathbb{R} from [48]. These are special solutions of the Szegő equation of the form $V(t, x) = e^{-i\omega t} V_0(x - ct)$.

Proposition 2.2 ([48], Traveling waves of the Szegő equation on \mathbb{R}). *A function $u \in C(\mathbb{R}; H_+^{\frac{1}{2}})$ is a traveling wave solution of the Szegő equation on \mathbb{R} if and only if there exist $\phi, a \in \mathbb{R}$ and $\alpha, p > 0$ such that*

$$V(t, x) = \frac{\alpha e^{i\phi} e^{-i\omega t}}{x - ct + a + ip} \quad \text{for all } t \in \mathbb{R},$$

where

$$c := \frac{\alpha^2}{2p} \quad \text{and} \quad \omega := \frac{\alpha^2}{4p^2} \quad (2.1)$$

Next, we introduce a basic construction of two traveling waves of the Szegő equation whose distance between each other exhibits a relative growth in time. This basic construction plays an essential role in the proof of Theorem 1.3 (i). Moreover, this immediately yields the failure of local uniform continuity of the solution map for the Szegő equation in $H_+^s(\mathbb{R})$, $s \in [0, \frac{1}{2})$, as shown in Corollary 2.4 below.

Proposition 2.3 (Basic construction for the Szegő equation). *Let $s > -\frac{1}{2}$ and $\delta > 0$. Given $0 < \varepsilon \ll 1$, there exist global solutions $\tilde{V}_1^\varepsilon, \tilde{V}_2^\varepsilon \in C(\mathbb{R}; H_+^s(\mathbb{R}))$ of (1.3) such that*

$$\|\tilde{V}_1^\varepsilon(0)\|_{H_+^s} + \|\tilde{V}_2^\varepsilon(0)\|_{H_+^s} \lesssim \varepsilon, \quad \|\tilde{V}_1^\varepsilon(0) - \tilde{V}_2^\varepsilon(0)\|_{H_+^s} \sim \varepsilon |\log \varepsilon|^{-\frac{1}{2}}, \quad (2.2)$$

and

$$\|\tilde{V}_1^\varepsilon(t) - \tilde{V}_2^\varepsilon(t)\|_{H_+^s} \gtrsim \varepsilon \quad (2.3)$$

for all $t \geq \frac{\delta}{\varepsilon^2} |\log \varepsilon|$.

The role of the parameter δ will be clear in the proof of Proposition 2.7, where it is chosen sufficiently small so as to apply the approximation result of Proposition 2.6.

Proof of Proposition 2.3. We choose \tilde{V}_j^ε to be the following traveling waves of the Szegő equation on \mathbb{R} :

$$\tilde{V}_j^\varepsilon(t, x) := \frac{\alpha_j e^{-i\omega_j t}}{x - c_j t + ip}, \quad (2.4)$$

where

$$p := 1, \quad \alpha_1 := \varepsilon, \quad \alpha_2 := \varepsilon(1 + |\log \varepsilon|^{-\frac{1}{2}}). \quad (2.5)$$

By (2.1), notice that we have

$$c_1 = \frac{\varepsilon^2}{2}, \quad c_2 = \frac{\varepsilon^2(1 + 2|\log \varepsilon|^{-\frac{1}{2}} + |\log \varepsilon|^{-1})}{2}, \quad c_2 - c_1 = \varepsilon^2 |\log \varepsilon|^{-\frac{1}{2}} (1 + o(1)). \quad (2.6)$$

Then, by Lemma 2.1, it follows that

$$\|\tilde{V}_1^\varepsilon(0) - \tilde{V}_2^\varepsilon(0)\|_{H_+^s} = (\alpha_2 - \alpha_1) \left\| \frac{1}{x + ip} \right\|_{H_+^s} \sim \varepsilon |\log \varepsilon|^{-\frac{1}{2}},$$

$\|\tilde{V}_1^\varepsilon(0)\|_{H_+^s} + \|\tilde{V}_2^\varepsilon(0)\|_{H_+^s} \lesssim \varepsilon$, and thus (2.2) is satisfied. Then, using again Lemma 2.1, we have

$$\begin{aligned} \|\tilde{V}_1^\varepsilon(t) - \tilde{V}_2^\varepsilon(t)\|_{\dot{H}_+^s}^2 &= \|\tilde{V}_1^\varepsilon(t)\|_{\dot{H}_+^s}^2 + \|\tilde{V}_2^\varepsilon(t)\|_{\dot{H}_+^s}^2 - 4\pi\alpha_1\alpha_2 \operatorname{Re} \left(e^{i(\omega_2 - \omega_1)t} \int_0^\infty \xi^{2s} e^{-2p\xi} e^{i\xi(c_2 - c_1)t} d\xi \right) \\ &= \varepsilon^2 \frac{\pi\Gamma(2s+1)}{2^{2s-1}} (1 + o(1)) - A, \end{aligned} \quad (2.7)$$

where

$$A := 4\pi\alpha_1\alpha_2 \operatorname{Re} \left(e^{i(\omega_2 - \omega_1)t} \int_0^\infty \xi^{2s} e^{-2p\xi} e^{i\xi(c_2 - c_1)t} d\xi \right).$$

Next, we show that $|A| \ll \varepsilon^2$. This comes down to finding an expression for $\int_0^\infty x^a e^{-\lambda x} dx$ for $a > -1$ and $\operatorname{Re} \lambda > 0$. If $\lambda \in \mathbb{R}_+$, then by a change of variables we have

$$\int_0^\infty x^a e^{-\lambda x} dx = \frac{1}{\lambda^{a+1}} \int_0^\infty x^a e^{-x} dx = \frac{\Gamma(a+1)}{\lambda^{a+1}}.$$

Since both $\int_0^\infty x^a e^{-\lambda x} dx$ and $\frac{\Gamma(a+1)}{\lambda^{a+1}}$ are holomorphic in λ for $\operatorname{Re} \lambda > 0$ and since they coincide on \mathbb{R}_+ , it follows that they coincide on $\{\lambda : \operatorname{Re} \lambda > 0\}$. Therefore,

$$\int_0^\infty \xi^{2s} e^{-\xi(2p - i(c_2 - c_1)t)} d\xi = \frac{\Gamma(2s+1)}{(2p - i(c_2 - c_1)t)^{2s+1}}.$$

Taking $t \geq \frac{\delta}{\varepsilon^2} |\log \varepsilon|$ and using (2.6), we get that

$$|A| \lesssim \frac{\varepsilon^2}{|2 - i(c_2 - c_1)t|^{2s+1}} \lesssim \frac{\varepsilon^2}{(\delta |\log \varepsilon|^{\frac{1}{2}})^{2s+1}} \ll \varepsilon^2.$$

Combining this with (2.7), we thus obtain that

$$\|\tilde{V}_1^\varepsilon(t) - \tilde{V}_2^\varepsilon(t)\|_{\dot{H}_+^s} \sim_s \varepsilon,$$

which yields (2.3). \square

Corollary 2.4 (Failure of local uniform continuity for SZ in $H_+^s(\mathbb{R})$, $s \in [0, \frac{1}{2})$). *Let $s \in [0, \frac{1}{2})$. Given $0 < \varepsilon \ll 1$, there exist global solutions $V_1^\varepsilon, V_2^\varepsilon \in C(\mathbb{R}; H_+^s(\mathbb{R}))$ of (1.3) such that $\|V_1^\varepsilon(0)\|_{H_+^s} + \|V_2^\varepsilon(0)\|_{H_+^s} \lesssim 1$,*

$$\lim_{\varepsilon \rightarrow 0} \|V_1^\varepsilon(0) - V_2^\varepsilon(0)\|_{H_+^s} = 0,$$

and

$$\liminf_{\varepsilon \rightarrow 0} \|V_1^\varepsilon - V_2^\varepsilon\|_{L^\infty([0, T]; H_+^s)} \gtrsim 1 \quad \text{for all } T > 0.$$

Proof. First, we consider the case $s \in (0, \frac{1}{2})$. We define

$$V_j^\varepsilon(t, x) := (\tilde{V}_j^\varepsilon)_\lambda(t, x) = \lambda^{\frac{1}{2}} \tilde{V}_j^\varepsilon(\lambda t, \lambda x),$$

where \tilde{V}_j^ε are as in Proposition 2.3 with $\delta = 1$, $j = 1, 2$, and $\lambda = \varepsilon^{-\frac{1}{s}}$. The functions V_j^ε are still solutions of the Szegő equation. Notice that $\|V_1^\varepsilon(t) - V_2^\varepsilon(t)\|_{L_+^2} = \|\tilde{V}_1^\varepsilon(\lambda t) - \tilde{V}_2^\varepsilon(\lambda t)\|_{L_+^2}$,

while $\|V_1^\varepsilon(t) - V_2^\varepsilon(t)\|_{\dot{H}_+^s} = \varepsilon^{-1} \|\tilde{V}_1^\varepsilon(\lambda t) - \tilde{V}_2^\varepsilon(\lambda t)\|_{\dot{H}_+^s}$ for all $t \in \mathbb{R}$. Combining these with (2.2) and (2.3), we obtain

$$\begin{aligned} \|V_1^\varepsilon(0)\|_{H_+^s} + \|V_2^\varepsilon(0)\|_{H_+^s} &\lesssim_s 1 \\ \|V_1^\varepsilon(0) - V_2^\varepsilon(0)\|_{H_+^s} &\sim_s |\log \varepsilon|^{-\frac{1}{2}} \ll 1, \\ \|V_1^\varepsilon(t) - V_2^\varepsilon(t)\|_{\dot{H}_+^s} &\sim_s 1 \end{aligned}$$

for $t \geq \varepsilon^{\frac{1}{s}-2} |\log \varepsilon|$. Noting that $\varepsilon^{\frac{1}{s}-2} |\log \varepsilon| \rightarrow 0$ as $\varepsilon \rightarrow 0$ precisely when $0 < s < \frac{1}{2}$ concludes the proof in the case $s \in (0, \frac{1}{2})$.

Next, we turn to the case $s = 0$. Because of the singularity at $s = 0$ in the scaling $V_j^\varepsilon = (\tilde{V}_j^\varepsilon)_\lambda$, $\lambda = \varepsilon^{-\frac{1}{s}}$, we can no longer use the above approach. Instead, here we consider

$$V_j^\varepsilon(t, x) := \frac{\alpha_j e^{-i\omega_j t}}{x - c_j t + ip},$$

where

$$p := \varepsilon, \quad \alpha_1 := \varepsilon^{\frac{1}{2}}, \quad \alpha_2 := \varepsilon^{\frac{1}{2}} (1 + |\log \varepsilon|^{-\frac{1}{2}}).$$

By (2.1), we notice that

$$c_1 = \frac{1}{2}, \quad c_2 = \frac{(1 + |\log \varepsilon|^{-\frac{1}{2}})^2}{2}, \quad c_2 - c_1 = |\log \varepsilon|^{-\frac{1}{2}} (1 + o(1)).$$

By Lemma 2.1, we have that $\|V_1^\varepsilon(0)\|_{L_+^2} = \sqrt{\pi}$, $\|V_2^\varepsilon(0)\|_{L_+^2} = \sqrt{\pi} (1 + |\log \varepsilon|^{-\frac{1}{2}})$, and

$$\|V_2^\varepsilon(0) - V_1^\varepsilon(0)\|_{L_+^2} = \varepsilon^{\frac{1}{2}} |\log \varepsilon|^{-\frac{1}{2}} \left\| \frac{1}{x + i\varepsilon} \right\|_{L_+^2} \sim |\log \varepsilon|^{-\frac{1}{2}} \ll 1.$$

Moreover,

$$\|V_1^\varepsilon(t) - V_2^\varepsilon(t)\|_{L_+^2}^2 = \|V_1^\varepsilon(t)\|_{L_+^2}^2 + \|V_2^\varepsilon(t)\|_{L_+^2}^2 - A' \geq 2\pi - A',$$

where

$$A' := 4\pi\alpha_1\alpha_2 \operatorname{Re} \left(e^{i(\omega_2 - \omega_1)t} \int_0^\infty e^{-2p\xi} e^{i\xi(c_2 - c_1)t} d\xi \right).$$

Arguing as in the proof of Proposition 2.3, we have for $t \geq \varepsilon |\log \varepsilon|$ that

$$|A'| \lesssim \frac{\varepsilon}{|2\varepsilon - i(c_2 - c_1)t|} \lesssim |\log \varepsilon|^{-\frac{1}{2}} \ll 1.$$

Therefore, for $t \geq \varepsilon |\log \varepsilon|$, we have indeed that

$$\|V_1^\varepsilon(t) - V_2^\varepsilon(t)\|_{L_+^2} \sim 1.$$

□

Remark 2.5. In the proof of Corollary 2.4, instead of using slightly different approaches for the cases $s \in (0, \frac{1}{2})$ and $s = 0$, one can choose to work, for all $s \in [0, \frac{1}{2})$, with the following traveling waves of the Szegő equation:

$$V_j^\varepsilon(t, x) := \frac{\alpha_j e^{-i\omega_j t}}{x - c_j t + ip}, \quad \text{where } p := \varepsilon, \quad \alpha_1 := \varepsilon^{s+\frac{1}{2}}, \quad \alpha_2 := \varepsilon^{s+\frac{1}{2}} (1 + |\log \varepsilon|^{-\frac{1}{2}}).$$

For $s \in (0, \frac{1}{2})$, however, we preferred to use the basic construction in Proposition 2.3 together with a scaling argument, as a preamble to our proof of the failure of local uniform continuity for NHW in $H^s(\mathbb{R})$, $s \in (0, \frac{1}{2})$, in Proposition 2.7.

Next, we recall the result from [49] on long time approximation of solutions of the NHW equation (1.2) by solutions of the Szegő model (1.3).

Proposition 2.6 ([49]). *Let $0 < \varepsilon \ll 1$, $\delta > 0$ sufficiently small, $s > \frac{1}{2}$, and $f \in H_+^s(\mathbb{R})$. Let $\mu \in \{-1, +1\}$ and $u_\mu \in C(\mathbb{R}; H^s)$ be the solution of (1.2) with initial data $u_\mu(0) = \varepsilon f$.*

Let $V \in C(\mathbb{R}; H_+^s)$ be the solution of the Szegő equation (1.3) with the same initial data $V(0) = \varepsilon f$. Assume that $\|V(t)\|_{H^s} \leq C\varepsilon$ for all $0 \leq t \leq \frac{\delta}{\varepsilon^2} |\log \varepsilon|$. Then, for any $0 \leq t \leq \frac{\delta}{\varepsilon^2} |\log \varepsilon|$, the following holds:

$$\|u_\mu(t, \cdot) - V(t, \cdot - t)\|_{H^s} \leq C_* \varepsilon^{2-C_0\delta},$$

where $C_0 > 0$ is an absolute constant and $C_* = C_*(\|f\|_{H^{\frac{1}{2}}})$.

With this approximation result at hand, we are now ready to state and prove the failure of uniform continuity of the solution map of (1.2) on bounded sets of $H^s(\mathbb{R})$, $s \in (0, \frac{1}{2})$.

Proposition 2.7 (Failure of local uniform continuity for NHW in $H^s(\mathbb{R})$, $s \in (0, \frac{1}{2})$). *Let $s \in (0, \frac{1}{2})$ and $\mu \in \{-1, +1\}$. Given $0 < \varepsilon \ll 1$, there exist global solutions u_1^ε and u_2^ε of (1.2) such that $\|u_1^\varepsilon(0)\|_{H^s} \lesssim 1$, $\|u_2^\varepsilon(0)\|_{H^s} \lesssim 1$,*

$$\lim_{\varepsilon \rightarrow 0} \|u_1^\varepsilon(0) - u_2^\varepsilon(0)\|_{H^s} = 0,$$

and

$$\liminf_{\varepsilon \rightarrow 0} \|u_1^\varepsilon - u_2^\varepsilon\|_{L^\infty([0, T]; H^s)} \gtrsim 1 \quad \text{for all } T > 0.$$

Proof of Proposition 2.7. We set $\tilde{v}_j^\varepsilon(t, x) := e^{-i|D|t} \tilde{V}_j^\varepsilon(t, x) = \tilde{V}_j^\varepsilon(t, x - t)$, $j = 1, 2$, where \tilde{V}_j^ε are the traveling waves of the Szegő equation introduced in (2.4) and (2.5). By Proposition 2.3, it follows that

$$\|\tilde{v}_1^\varepsilon(0) - \tilde{v}_2^\varepsilon(0)\|_{H^s} \sim \varepsilon |\log \varepsilon|^{-\frac{1}{2}} \quad (2.8)$$

and

$$\|\tilde{v}_1^\varepsilon(t) - \tilde{v}_2^\varepsilon(t)\|_{H^s} \gtrsim \varepsilon \quad (2.9)$$

for all $t \geq \frac{\delta}{\varepsilon^2} |\log \varepsilon|$, where $\delta > 0$ is a small real number to be chosen later.

Next, we denote by \tilde{u}_j^ε , $j = 1, 2$, the smooth solutions of (1.2) with initial data

$$\tilde{u}_j^\varepsilon(0, x) := \tilde{v}_j^\varepsilon(0, x) = \frac{\alpha_j}{x + i},$$

where α_j are as in (2.5). Note that the initial conditions $\tilde{u}_j^\varepsilon(0)$ are sufficiently smooth and small to guarantee the global existence of the corresponding solutions of NHW in both the defocusing and focusing cases.

By Lemma 2.1 we have $\|\tilde{u}_j^\varepsilon(0)\|_{H^s} \lesssim \varepsilon$ and $\|\tilde{v}_j^\varepsilon(t)\|_{H^s} \sim \varepsilon$ for all $t \in \mathbb{R}$. Thus, applying Proposition 2.6, it then follows that for all $0 \leq t \leq \frac{\delta}{\varepsilon^2} |\log \varepsilon|$ we have

$$\|\tilde{u}_j^\varepsilon(t) - \tilde{v}_j^\varepsilon(t)\|_{H^s} \lesssim \varepsilon^{2-C_0\delta}, \quad (2.10)$$

where $C_0 > 0$ is an absolute constant. Combining (2.8), (2.9), and (2.10), we then obtain that:

$$\begin{aligned} \|\tilde{u}_1^\varepsilon(0) - \tilde{u}_2^\varepsilon(0)\|_{H^s} &\sim \varepsilon |\log \varepsilon|^{-\frac{1}{2}} \\ \left\| (\tilde{u}_1^\varepsilon - \tilde{u}_2^\varepsilon) \left(\frac{\delta}{2\varepsilon^2} |\log \varepsilon| \right) \right\|_{H^s} &\gtrsim \varepsilon, \end{aligned}$$

provided that δ is so small that $2 - C_0\delta > 1$. We set $u_j^\varepsilon := (\tilde{u}_j^\varepsilon)_\lambda$ with $\lambda = \varepsilon^{-\frac{1}{s}}$, and observe that u_j^ε is also a solution of NHW. Then, we obtain as in the proof of Corollary 2.4 that

$$\begin{aligned} \|u_1^\varepsilon(0)\|_{H^s} + \|u_2^\varepsilon(0)\|_{H^s} &\lesssim 1, \\ \|u_1^\varepsilon(0) - u_2^\varepsilon(0)\|_{H^s} &\sim |\log \varepsilon|^{-\frac{1}{2}} \ll 1, \\ \|u_1^\varepsilon(T_\varepsilon) - u_2^\varepsilon(T_\varepsilon)\|_{H^s} &\gtrsim 1, \end{aligned}$$

where $T_\varepsilon := \delta \varepsilon^{\frac{1}{s}-2} |\log \varepsilon| \ll 1$. This concludes the proof of the proposition. \square

Remark 2.8. Setting $v_j^\varepsilon := (\tilde{v}_j^\varepsilon)_\lambda$ with $\lambda = \varepsilon^{-\frac{1}{s}}$ (where \tilde{v}_j^ε were defined in the proof of Proposition 2.7) and $\tilde{\varepsilon} := \varepsilon^{\frac{1}{s}}$, we have that

$$v_j^\varepsilon(t, x) = \frac{\tilde{\alpha}_j e^{-i\tilde{\omega}_j t}}{x - (1 + c_j)t + i\tilde{\varepsilon}},$$

where

$$\tilde{\alpha}_1 = \tilde{\varepsilon}^{s+\frac{1}{2}}, \quad \tilde{\alpha}_2 = \tilde{\varepsilon}^{s+\frac{1}{2}}(1 + C(s)|\log \tilde{\varepsilon}|^{-\frac{1}{2}}), \quad \tilde{\omega}_j = \omega_j \tilde{\varepsilon}^{-1}, \quad j = 1, 2.$$

Then, we can reformulate (2.10) as

$$\|u_j^\varepsilon(t) - v_j^\varepsilon(t)\|_{H^s} \lesssim \tilde{\varepsilon}^{s(1-C_0\delta)}$$

for all $0 \leq t \lesssim \delta \tilde{\varepsilon}^{1-2s} |\log \tilde{\varepsilon}|$ and $\delta > 0$ sufficiently small.

We remark that v_j^ε are (translated) traveling waves of the Szegő equation on \mathbb{R} analogous to the traveling waves considered in [24] for the Szegő equation on \mathbb{T} . In [24, Proposition 3.1], the difference $\|u_j^\varepsilon - v_j^\varepsilon\|_{H^s}$ is bounded above by $\tilde{\varepsilon}^{s-\frac{1}{4}}$. This explains why the failure of local uniform continuity for periodic NHW obtained in [24] occurs in $H^s(\mathbb{T})$ with the restriction on the regularity $s \in (\frac{1}{4}, \frac{1}{2})$. In our context, the upper bound $\tilde{\varepsilon}^{s(1-C_0\delta)}$ allows for the wider regularity range $s \in (0, \frac{1}{2})$.

Remark 2.9. The scaling used in the proof of Proposition 2.7, $u_j^\varepsilon := (\tilde{u}_j^\varepsilon)_\lambda$ with $\lambda = \varepsilon^{-\frac{1}{s}}$, is only defined for $s \neq 0$. Therefore, even though the solution map of the Szegő equation on \mathbb{R} fails to be locally uniformly continuous in $L^2(\mathbb{R})$, as proved in Corollary 2.4 above, one cannot use the approximation/scaling argument in Proposition 2.7 to deduce the same behavior for the solution map of NHW on $L^2(\mathbb{R})$.

3. FAILURE OF LOCAL UNIFORM CONTINUITY OF THE SOLUTION MAP OF THE FOCUSING NHW IN $L^2(\mathbb{R})$

In this section, we prove Theorem 1.3 (ii). More precisely, we consider the focusing half-wave equation (1.2) (with $\mu = -1$) and show that its solution map fails to be uniformly

continuous on bounded sets of $L^2(\mathbb{R})$. As noted above in Remark 2.9, the approximation/scaling argument from Proposition 2.7 can no longer be used in the case of $L^2(\mathbb{R})$. Consequently, we consider a different approach. Namely, instead of working with solutions of NHW that can be approximated by traveling waves of the Szegő equation, we work directly with traveling waves of the focusing NHW.

Proposition 3.1 (Failure of local uniform continuity for focusing NHW in $L^2(\mathbb{R})$). *Given $0 < \varepsilon \ll 1$, there exist global solutions u_1^ε and u_2^ε of the focusing half-wave equation (1.2) with $\mu = -1$ such that $\|u_1^\varepsilon(0)\|_{L^2} \lesssim 1$, $\|u_2^\varepsilon(0)\|_{L^2} \lesssim 1$,*

$$\lim_{\varepsilon \rightarrow 0} \|u_1^\varepsilon(0) - u_2^\varepsilon(0)\|_{L^2} = 0,$$

and

$$\liminf_{\varepsilon \rightarrow 0} \|u_1^\varepsilon - u_2^\varepsilon\|_{L^\infty([0,T];L^2)} \gtrsim 1 \quad \text{for all } T > 0.$$

The solutions u_1^ε , u_2^ε , that we use to prove Proposition 3.1 are conveniently rescaled versions of traveling waves of the focusing NHW. In [38], Krieger, Lenzmann, and Raphaël showed that for any $-1 < \beta < 1$, the focusing cubic half-wave equation on \mathbb{R} possesses a traveling wave solution $u_\beta(t, x) := Q_\beta \left(\frac{x - \beta t}{1 - \beta} \right) e^{it}$, where $Q_\beta \in H^{\frac{1}{2}}(\mathbb{R})$ satisfies

$$\frac{|D| - \beta D}{1 - \beta} Q_\beta + Q_\beta = |Q_\beta|^2 Q_\beta.$$

The crucial element in the proof of Proposition 3.1 is the use of certain properties of Q_β that we recall below from [27].

Lemma 3.2 ([27], Properties of Q_β). *There exists $0 < \beta_* < 1$ such that for all $\beta, \tilde{\beta} \in (\beta_*, 1)$, the following hold:*

$$\|Q_\beta - Q_{\tilde{\beta}}\|_{H^{\frac{1}{2}}} \leq C \frac{|\beta - \tilde{\beta}|}{\min(1 - \beta, 1 - \tilde{\beta})} \quad (3.1)$$

$$\|x \partial_x Q_\beta\|_{L^2} \leq C \quad (3.2)$$

$$|Q_\beta(x)| \leq \frac{C}{\langle x \rangle (1 + (1 - \beta) \langle x \rangle)}, \quad (3.3)$$

where $C > 0$ is an absolute constant (independent of β , $\tilde{\beta}$) and $\langle x \rangle := \sqrt{1 + x^2}$.

Moreover, given $\beta \in (\beta_*, 1)$, there exist constants $x(\beta) \in \mathbb{R}$ and $\gamma \in \mathbb{T}$ such that, up to a subsequence,

$$\|Q_\beta(x - x(\beta)) - e^{i\gamma} Q^+(x)\|_{H^{\frac{1}{2}}} \leq C_1 (1 - \beta)^{\frac{1}{8}}, \quad (3.4)$$

where $Q^+(x) = \frac{2}{2x+i}$ and $C_1 > 2$ is an absolute constant. In particular, $Q_\beta(\cdot - x(\beta)) \rightarrow e^{i\gamma} Q^+$ in $H^{\frac{1}{2}}(\mathbb{R})$ as $\beta \rightarrow 1$.

The proof of Lemma 3.2 is lengthy and does not constitute the object of this paper. Therefore, we decided to omit it here and we refer the readers to [27] for details.

The following lemma is another useful tool in the proof of Proposition 3.1.

Lemma 3.3 ([27], Auxiliary lemma). *Let $\beta \in (0, 1)$. Then, the following holds:*

$$\int_{\mathbb{R}} \frac{1}{\langle x-y \rangle (1 + (1-\beta)\langle x-y \rangle)} \cdot \frac{1}{\langle y \rangle (1 + (1-\beta)\langle y \rangle)} dy \lesssim \frac{|\log(1-\beta)|}{\langle x \rangle (1 + (1-\beta)\langle x \rangle)}.$$

The proof of Lemma 3.3 is elementary and consists of analyzing separately the following three regions of integration: (i) $|y| \geq 2|x|$, (ii) $|y| \leq 2|x|$ and $|x-y| \geq \frac{|x|}{2}$, (iii) $|y| \leq 2|x|$ and $|x-y| \leq \frac{|x|}{2}$. Details can be found in the appendix of [27].

Proof of Proposition 3.1. We choose $0 < \varepsilon \ll 1$, $c_0 \in \left(0, 1 - \max\left(\beta_*, 1 - \left(\frac{\pi}{2C_1^2}\right)^4\right)\right)$, where C_1 is as in (3.4), and β_1 and β_2 such that

$$0 < \max\left(\beta_*, 1 - \left(\frac{\pi}{2C_1^2}\right)^4\right) < \beta_1 < \beta_2 < 1 - c_0, \quad \varepsilon^{\frac{4}{3}}(1 - \beta_2) < \beta_2 - \beta_1 < \varepsilon(1 - \beta_2). \quad (3.5)$$

In other words, the speeds β_1 and β_2 are sufficiently close to 1, but away from 1, and ε -close to each other.

We start by estimating the difference of the initial data of the solutions u_{β_1} and u_{β_2} . By (3.1), (3.2), and (3.5), it follows that

$$\begin{aligned} \|u_{\beta_1}(0, x) - u_{\beta_2}(0, x)\|_{L^2} &= \left\| Q_{\beta_1}\left(\frac{x}{1-\beta_1}\right) - Q_{\beta_2}\left(\frac{x}{1-\beta_2}\right) \right\|_{L^2} \\ &= \sqrt{1-\beta_1} \left\| Q_{\beta_1}(x) - Q_{\beta_2}\left(\frac{1-\beta_1}{1-\beta_2}x\right) \right\|_{L^2} \\ &\leq \sqrt{1-\beta_1} \left(\|Q_{\beta_1}(x) - Q_{\beta_2}(x)\|_{L^2} + \left\| Q_{\beta_2}(x) - Q_{\beta_2}\left(\frac{1-\beta_1}{1-\beta_2}x\right) \right\|_{L^2} \right) \\ &\lesssim \sqrt{1-\beta_1} \frac{\beta_2 - \beta_1}{1-\beta_2} \left(1 + \sup_{c \in [1, \frac{1-\beta_1}{1-\beta_2}]} \|x(\partial_x Q_{\beta})(cx)\|_{L^2} \right) \lesssim \sqrt{1-\beta_1} \frac{\beta_2 - \beta_1}{1-\beta_2} \\ &\lesssim \frac{\beta_2 - \beta_1}{\sqrt{1-\beta_2}} + \frac{(\beta_2 - \beta_1)^{\frac{3}{2}}}{1-\beta_2} \lesssim \varepsilon \sqrt{1-\beta_2} \lesssim \varepsilon. \end{aligned} \quad (3.6)$$

Next, we estimate the difference of the solutions at time t .

$$\begin{aligned} \|u_{\beta_1}(t, x) - u_{\beta_2}(t, x)\|_{L^2}^2 &= \left\| Q_{\beta_1}\left(\frac{x - \beta_1 t}{1-\beta_1}\right) - Q_{\beta_2}\left(\frac{x - \beta_2 t}{1-\beta_2}\right) \right\|_{L^2}^2 \\ &= \left\| Q_{\beta_1}\left(\frac{x - \beta_1 t}{1-\beta_1}\right) \right\|_{L^2}^2 + \left\| Q_{\beta_2}\left(\frac{x - \beta_2 t}{1-\beta_2}\right) \right\|_{L^2}^2 - 2 \operatorname{Re} \int_{\mathbb{R}} \overline{Q_{\beta_1}\left(\frac{x - \beta_1 t}{1-\beta_1}\right)} Q_{\beta_2}\left(\frac{x - \beta_2 t}{1-\beta_2}\right) dx \\ &= (1-\beta_1) \|Q_{\beta_1}\|_{L^2}^2 + (1-\beta_2) \|Q_{\beta_2}\|_{L^2}^2 - B, \end{aligned} \quad (3.7)$$

where

$$B := 2(1-\beta_1) \operatorname{Re} \int_{\mathbb{R}} \overline{Q_{\beta_1}(y_1)} Q_{\beta_2}\left(\frac{1-\beta_1}{1-\beta_2}y_1 + \frac{\beta_1 - \beta_2}{1-\beta_2}t\right) dy_1.$$

We first estimate $|B|$. By (3.3) and setting $y_2 := \frac{1-\beta_1}{1-\beta_2}y_1 + \frac{\beta_1 - \beta_2}{1-\beta_2}t$, we have

$$|B| \lesssim (1-\beta_1) \int_{\mathbb{R}} \frac{1}{\langle y_1 \rangle (1 + (1-\beta_1)\langle y_1 \rangle)} \cdot \frac{1}{\langle y_2 \rangle (1 + (1-\beta_2)\langle y_2 \rangle)} dy_1. \quad (3.8)$$

Using $\beta_2 - \beta_1 < \varepsilon(1 - \beta_2) < \varepsilon(1 - \beta_1)$, we notice that $\frac{1-\beta_2}{1-\beta_1} = 1 - \frac{\beta_2-\beta_1}{1-\beta_1} > 1 - \varepsilon$ and thus,

$$\begin{aligned} \frac{1}{\langle y_2 \rangle} &\sim \frac{1}{1 + |y_2|} \sim \frac{1 - \beta_2}{1 - \beta_1} \cdot \frac{1}{\frac{1-\beta_2}{1-\beta_1} + |y_1 + \frac{\beta_1-\beta_2}{1-\beta_1}t|} \lesssim \frac{1 - \beta_2}{1 - \beta_1} \cdot \frac{1}{1 - \varepsilon + |y_1 + \frac{\beta_1-\beta_2}{1-\beta_1}t|} \\ &\lesssim \frac{1 - \beta_2}{1 - \beta_1} \cdot \frac{1}{\langle y_1 + \frac{\beta_1-\beta_2}{1-\beta_1}t \rangle}. \end{aligned} \quad (3.9)$$

Similarly, we have

$$\frac{1}{1 + (1 - \beta_2)\langle y_2 \rangle} \sim \frac{1}{1 + (1 - \beta_1) \left(\frac{1-\beta_2}{1-\beta_1} + |y_1 + \frac{\beta_1-\beta_2}{1-\beta_1}t| \right)} \lesssim \frac{1}{1 + (1 - \beta_1)\langle y_1 + \frac{\beta_1-\beta_2}{1-\beta_1}t \rangle}. \quad (3.10)$$

Therefore, by (3.8), (3.9), (3.10), and using Lemma 3.3, we obtain that

$$\begin{aligned} |B| &\lesssim (1 - \beta_2) \int_{\mathbb{R}} \frac{1}{\langle y_1 \rangle (1 + (1 - \beta_1)\langle y_1 \rangle)} \cdot \frac{1}{\langle y_1 + \frac{\beta_1-\beta_2}{1-\beta_1}t \rangle \left(1 + (1 - \beta_1)\langle y_1 + \frac{\beta_1-\beta_2}{1-\beta_1}t \rangle \right)} dy_1 \\ &\lesssim \frac{(1 - \beta_2)|\log(1 - \beta_1)|}{\langle \frac{\beta_1-\beta_2}{1-\beta_1}t \rangle \left(1 + (1 - \beta_1)\langle \frac{\beta_1-\beta_2}{1-\beta_1}t \rangle \right)} \lesssim \frac{(1 - \beta_1)(1 - \beta_2)|\log(1 - \beta_1)|}{((\beta_2 - \beta_1)t)^2}. \end{aligned}$$

By choosing $t \geq \frac{1}{\sqrt{\varepsilon}(\beta_2 - \beta_1)}$ and using $(1 - \beta_1)|\log(1 - \beta_1)| \lesssim 1$, it follows that

$$|B| \ll \varepsilon(1 - \beta_2) < \varepsilon(1 - \beta_1). \quad (3.11)$$

On the other hand, by (3.4), using $\|Q^+\|_{L^2} = \sqrt{2\pi}$ and $1 - \beta_2 < 1 - \beta_1 < \left(\frac{\pi}{2C_1^2}\right)^4$, we have

$$\|Q_{\beta_j}\|_{L^2} \geq \|Q^+\|_{L^2} - \|Q_{\beta_j}(\cdot - x(\beta_j)) - e^{i\gamma}Q^+\|_{L^2} \geq \sqrt{2\pi} - C_1(1 - \beta_1)^{\frac{1}{8}} \geq \sqrt{\frac{\pi}{2}} \quad (3.12)$$

for $j = 1, 2$. Then, combining (3.7), (3.11), and (3.12), we obtain for $t \geq \frac{1}{\sqrt{\varepsilon}(\beta_2 - \beta_1)}$ that

$$\|u_{\beta_1}(t) - u_{\beta_2}(t)\|_{L^2} \gtrsim \sqrt{1 - \beta_1} \gtrsim \sqrt{c_0}. \quad (3.13)$$

Finally, by considering the rescaled variants of u_{β_j} $u_j^\varepsilon(t, x) := \varepsilon^{-1}u_{\beta_j}(\varepsilon^{-2}t, \varepsilon^{-2}x)$ and using (3.4), (3.6), and (3.13), we obtain that $\|u_j^\varepsilon(0)\|_{L^2} = \sqrt{1 - \beta_j}\|Q_{\beta_j}\|_{L^2} \lesssim \sqrt{1 - \beta_j} \lesssim \sqrt{1 - \beta_*}$ for $j = 1, 2$, $\|u_1^\varepsilon(0) - u_2^\varepsilon(0)\|_{L^2} \lesssim \varepsilon$, and

$$\|u_1^\varepsilon(t) - u_2^\varepsilon(t)\|_{L^2} \gtrsim 1$$

for all $t \geq \frac{\varepsilon^{\frac{3}{2}}}{\beta_2 - \beta_1}$, and in particular for all $t \gtrsim \varepsilon^{\frac{1}{6}}$ (since $\beta_2 - \beta_1 > \varepsilon^{\frac{4}{3}}(1 - \beta_2)$). This completes the proof. \square

Remark 3.4. (i). For the defocusing nonlinear half-wave equation (1.2) with $\mu = 1$, traveling waves are not available and therefore the above approach cannot be used to prove failure of local uniform continuity in $L^2(\mathbb{R})$.

For the defocusing NHW, it is natural to attempt to use the approach of Christ, Colliander, and Tao from [16]. Namely, one would like to consider two solutions $u_j(t, x) = \phi_j(t, \nu x)$,

$j = 1, 2$, $0 < \nu \ll 1$, where ϕ_j satisfy:

$$\begin{cases} i\partial_t \phi_j - \nu |D| \phi_j = |\phi_j|^2 \phi_j \\ \phi_j(0, x) = a_j w(x), \end{cases}$$

with w a fixed Schwartz function, $a_1, a_2 \in [\frac{1}{2}, 2]$, and $|a_1 - a_2| \ll 1$. One can then approximate ϕ_j , for a long time $0 \leq t \leq c|\log \nu|^c$, by the solution $\phi_j^{(0)}$ of dispersionless NLS, $i\partial_t \phi_j^{(0)} = |\phi_j^{(0)}|^2 \phi_j^{(0)}$, with the same initial data $\phi_j^{(0)}(0, x) = \phi_j(0, x)$. For $\phi_j^{(0)}$ we have an explicit formula $\phi_j^{(0)}(x) = a_j w(x) e^{ia_j^2 t |w(x)|^2}$. Therefore, one can show without difficulty that $\|\phi_1^{(0)}(0) - \phi_2^{(0)}(0)\|_{L^2} \lesssim |a_1 - a_2| \ll 1$ and $\|\phi_1^{(0)}(t) - \phi_2^{(0)}(t)\|_{L^2} \gtrsim 1$ provided that $t \gg \frac{1}{|a_1 - a_2|}$. However, the time on which the approximation of ϕ_j by $\phi_j^{(0)}$ holds, does not seem to be sufficiently long to obtain the same statement for $u_1 - u_2$. More precisely, one only obtains $\frac{\|u_1(t) - u_2(t)\|_{L^2}}{\|u_1(0) - u_2(0)\|_{L^2}} \gtrsim \frac{1}{|a_1 - a_2|} \gg 1$ with $\|u_1(0) - u_2(0)\|_{L^2} \gg 1$.

In [16], in the case of the nonlinear Schrödinger equation below the scaling critical regularity, this issue is addressed by using the scaling and Galilean symmetries of NLS and, as a result, one obtains indeed failure of local uniform continuity. In our context, however, we are at the scaling critical regularity and, therefore, the scaling is not useful. Moreover, NHW does not have a Galilean symmetry, nor a Lorenz symmetry. Therefore, unless a new invariance is found for NHW, this approach does not seem viable. We note that in [9], the use of scaling was avoided for super-quintic NLS on a three dimensional manifold by working with highly localized initial data. The strategy of [9] can be applied to show failure of local uniform continuity for a defocusing super-cubic half-wave equation in $L^2(\mathbb{R})$, but not for the cubic NHW that we consider here.

(ii). As we have seen in the proof of Proposition 2.7, the approximation by the Szegő equation does not seem sufficient to decide on the failure of local uniform continuity for the defocusing NHW in $L^2(\mathbb{R})$ (at least, not for the examples of solutions considered in Proposition 2.7). It would be interesting to find a better approximation of the defocusing NHW that might provide us with a more accurate understanding of the dynamics.

4. FAILURE OF C^3 -SMOOTHNESS OF THE SOLUTION MAP OF NHW IN $L^2(\mathbb{R})$

In the previous section, we discussed the failure of local uniform continuity of the solution map of NHW in $L^2(\mathbb{R})$. In Proposition 3.1, we showed this for the focusing NHW. It remains, however, an open question in the case of the defocusing NHW. In this section, we prove a weaker form of ill-posedness in $L^2(\mathbb{R})$ for both the defocusing and focusing NHW. Namely, we show that the solution map of NHW fails to be C^3 -smooth in $L^2(\mathbb{R})$ (assuming that it is well defined as a mapping on $L^2(\mathbb{R})$).

Proposition 4.1 (Failure of C^3 -smoothness of the solution map of NHW in $L^2(\mathbb{R})$). *Let $\mu \in \{-1, 1\}$ and fix $0 < t \leq 1$. Denote the solution map of (1.2) by $\Phi(t) : u_0 \mapsto u(t)$. Assuming that $\Phi(t)$ is well-defined as a map acting on $L^2(\mathbb{R})$, it then follows that $\Phi(t)$ is not C^3 -smooth at $u_0 = 0$ in $L^2(\mathbb{R})$.*

Proof. If the solution map $\Phi(t)$ were to be C^3 -smooth at zero in $L^2(\mathbb{R})$, then there would exist $C > 0$ such that for all $f \in L^2(\mathbb{R})$:

$$\left\| \frac{d^3\Phi(t)(\delta f)}{d\delta^3} \Big|_{\delta=0} \right\|_{L^2} \leq C \|f\|_{L^2}^3.$$

In the following we show that such an estimate cannot hold with a constant independent of $f \in L^2(\mathbb{R})$.

By Duhamel's formula, we have

$$\Phi(t)(\delta f) = \delta e^{-it|D|} f - i \int_0^t e^{-i(t-t')|D|} |\Phi(t')(\delta f)|^2 \Phi(t')(\delta f) dt'.$$

In turn, $\frac{d\Phi(t)(\delta f)}{d\delta} \Big|_{\delta=0} = e^{-it|D|} f$, $\frac{d^2\Phi(t)(\delta f)}{d\delta^2} \Big|_{\delta=0} = 0$, and

$$\frac{d^3\Phi(t)(\delta f)}{d\delta^3} \Big|_{\delta=0} = -6i \int_0^t e^{-i(t-t')|D|} (|e^{-it'|D|} f|^2 e^{-it'|D|} f)(x) dt'.$$

To prove the failure of C^3 -smoothness of $\Phi(t)$ at zero in $L^2(\mathbb{R})$, we show that

$$\left\| \int_0^t e^{-i(t-t')|D|} (|e^{-it'|D|} f|^2 e^{-it'|D|} f)(x) dt' \right\|_{L^2} \leq C \|f\|_{L^2}^3 \quad (4.1)$$

cannot hold uniformly in $f \in L^2(\mathbb{R})$.

Consider $f_\varepsilon(x) = \frac{1}{x+i\varepsilon} \in L_+^2(\mathbb{R})$. By Lemma 2.1, we have that

$$\|f_\varepsilon\|_{L^2} = \frac{1}{\sqrt{2\varepsilon}}. \quad (4.2)$$

Then, noticing that $e^{-it|D|}\Pi_+g(x) = \Pi_+g(x-t)$, $e^{-it|D|}\Pi_-g(x) = \Pi_-g(x+t)$, for all $g \in L^2(\mathbb{R})$, along with $\Pi_+f_\varepsilon = f_\varepsilon$, it follows that

$$\begin{aligned} \int_0^t e^{-i(t-t')|D|} (|e^{-it'|D|} f_\varepsilon|^2 e^{-it'|D|} f_\varepsilon)(x) dt' &= \int_0^t e^{-i(t-t')|D|} (|f_\varepsilon|^2 f_\varepsilon)(x-t') dt' \\ &= \int_0^t (\Pi_+(|f_\varepsilon|^2 f_\varepsilon)(x-t) + \Pi_- (|f_\varepsilon|^2 f_\varepsilon)(x+t-2t')) dt' \\ &= t\Pi_+(|f_\varepsilon|^2 f_\varepsilon)(x-t) + \int_0^t \Pi_- (|f_\varepsilon|^2 f_\varepsilon)(x+t-2t') dt'. \end{aligned} \quad (4.3)$$

Decomposing into simple fractions gives

$$|f_\varepsilon|^2 f_\varepsilon(x) = \frac{1}{4\varepsilon^2} \cdot \frac{1}{x+i\varepsilon} - \frac{1}{2i\varepsilon} \cdot \frac{1}{(x+i\varepsilon)^2} - \frac{1}{4\varepsilon^2} \cdot \frac{1}{x-i\varepsilon},$$

and observe that the first two terms are supported on non-negative frequencies, while the last term is supported on negative frequencies. Thus,

$$\Pi_+(|f_\varepsilon|^2 f_\varepsilon)(x) = \frac{1}{4\varepsilon^2} f_\varepsilon(x) + \frac{1}{2i\varepsilon} \partial_x f_\varepsilon(x), \quad \Pi_- (|f_\varepsilon|^2 f_\varepsilon)(x) = -\frac{1}{4\varepsilon^2} \cdot \frac{1}{x-i\varepsilon}.$$

Therefore, using again Lemma 2.1, we obtain that

$$\|t\Pi_+(|f_\varepsilon|^2 f_\varepsilon)(x-t)\|_{L^2} = \frac{t\sqrt{5\pi}}{4\varepsilon^2\sqrt{\varepsilon}}. \quad (4.4)$$

and

$$\mathcal{F} \left(\int_0^t \frac{dt'}{x+t-2t'-i\varepsilon} dt' \right) (\xi) = 2\pi e^{it\xi} \frac{1-e^{-2it\xi}}{2\xi} e^{\varepsilon\xi} \mathbf{1}_{\xi \leq 0}.$$

Using $|\frac{1-e^{-2it\xi}}{2\xi}| \leq t$, we then have that

$$\begin{aligned} \left\| \int_0^t \Pi_- (|f_\varepsilon|^2 f_\varepsilon)(x+t-2t') dt' \right\|_{L_x^2} &= \frac{1}{4\varepsilon^2} \left\| \mathcal{F} \left(\int_0^t \frac{dt'}{x+t-2t'-i\varepsilon} \right) (\xi) \right\|_{L_\xi^2} \\ &\lesssim \frac{1}{\varepsilon^2} \left(t^2 + \int_{-\infty}^{-1} \frac{1}{\xi^2} d\xi \right)^{\frac{1}{2}} \lesssim \frac{1}{\varepsilon^2}. \end{aligned} \quad (4.5)$$

Combining (4.3), (4.4), and (4.5), and recalling that $0 < t \leq 1$ is fixed, we obtain that

$$\left\| \int_0^t e^{-i(t-t')|D|} (|e^{-it'|D}| f_\varepsilon|^2 e^{-it'|D|} f_\varepsilon)(x) dt' \right\|_{L_x^2} \gtrsim \frac{t}{\varepsilon^2 \sqrt{\varepsilon}} \quad (4.6)$$

for ε sufficiently small. By making ε tend to zero, it follows by (4.2) and (4.6) that there is no constant C for which (4.1) holds. Therefore, indeed, $\Phi(t)$ is not C^3 -smooth in $L^2(\mathbb{R})$. \square

Remark 4.2. (i) The proof of Proposition 4.1 can be easily adapted to show the failure of C^3 -smoothness of the solution map of NHW in $H^s(\mathbb{R})$ for any $s \in [0, \frac{1}{2})$.

(ii) A simplified variant of the proof of Proposition 4.1 yields the failure of C^3 -smoothness of the solution map of the Szegő equation in $H_+^s(\mathbb{R})$, $s \in [0, \frac{1}{2})$.

5. NORM INFLATION PROPERTY FOR NHW IN $H^s(\mathbb{R})$, $s < 0$

This section is dedicated to the proof of Theorem 1.3 (iv). Namely, we show that a H^s -norm inflation phenomenon occurs for certain solutions of NHW when $s < 0$.

The analysis in this section follows closely an argument developed by Kishimoto [36] in the context of the one-dimensional periodic cubic nonlinear Schrödinger equation (see also [32]). An important tool is the use of an algebra contained in $L^2(\mathbb{R})$. We choose this algebra to be the following (scaled) modulation space.

Definition 5.1. Given $A \geq 1$, let $I_A := [-\frac{A}{2}, \frac{A}{2})$. We define $M_A(\mathbb{R})$ to be the completion of $C^\infty(\mathbb{R})$ with respect to the norm:

$$\|f\|_{M_A} := \sum_{k \in A\mathbb{Z}} \|\hat{f}\|_{L^2(k+I_A)}. \quad (5.1)$$

Modulation spaces were introduced by Feichtinger in [20] and the basic theory of these spaces was established in [21, 22]. See also [3] for an application of modulation spaces to the local well-posedness theory of nonlinear dispersive PDEs. In the present paper, we only use the following two properties of the modulation space M_A .

Lemma 5.2 (Properties of the modulation space M_A). *Let $A \geq 1$.*

- (i) *There exists an absolute constant $C > 0$ such that $\|f\|_{L^2} \leq C\|f\|_{M_A}$ for all $f \in M_A$.*
- (ii) *There exists $C_2 > 0$ absolute constant such that for any $f, g \in M_A$ the following holds:*

$$\|fg\|_{M_A} \leq C_2 A^{\frac{1}{2}} \|f\|_{M_A} \|g\|_{M_A}. \quad (5.2)$$

The algebra property (ii) in Lemma 5.2 allows one to easily show that NHW is locally well-posed in M_A . Before stating this local well-posedness result, we set the following notations for $\phi \in M_A$:

$$\begin{aligned} U_1[\phi](t) &:= e^{-it|D|}\phi \\ U_k[\phi](t) &:= -i\mu \sum_{\substack{k_1, k_2, k_3 \geq 1 \\ k_1 + k_2 + k_3 = k}} \int_0^t e^{-i(t-\tau)|D|} (U_{k_1}[\phi] \overline{U_{k_2}[\phi]} U_{k_3}[\phi])(\tau) d\tau. \end{aligned} \quad (5.3)$$

Here, $U_k[\phi]$ is the sum of all the terms that contain exactly k factors $e^{-it|D|}\phi$ in the Picard iteration process of constructing a solution of (1.2) with initial condition ϕ . Note also that $U_k[\phi] \equiv 0$ for all even k .

Lemma 5.3 (Local well-posedness of NHW in M_A). *Let $\mu \in \{-1, 1\}$, $A \geq 1$, and $\phi \in M_A$. There exists a unique solution $u \in C([0, T_*]; M_A)$ of (1.2), where $T_* = C_3 A^{-1} \|\phi\|_{M_A}^{-2}$ and $C_3 > 0$ is an absolute constant. Moreover,*

$$u = \sum_{k=1}^{\infty} U_k[\phi], \quad (5.4)$$

where the series converges absolutely in $C([0, T_*]; M_A)$.

Proof. The proof is via a standard fixed point argument. We consider the operator

$$\Gamma u(t) := e^{-it|D|}\phi - i\mu \int_0^t e^{-i(t-\tau)|D|} |u|^2 u(\tau) d\tau.$$

By Lemma 5.2, we have for u in the ball $B(0, 2\|\phi\|_{M_A})$ in $C([0, T_*]; M_A)$ that

$$\sup_{t \in [0, T_*]} \|\Gamma u(t)\|_{M_A} \leq \|\phi\|_{M_A} + CT_* A \|u\|_{M_A}^3 \leq \|\phi\|_{M_A} (1 + 8CT_* A \|\phi\|_{M_A}^2) \leq 2\|\phi\|_{M_A}$$

provided that $T_* \leq (8C)^{-1} A^{-1} \|\phi\|_{M_A}^{-2}$. That is, Γ maps the ball $B(0, 2\|\phi\|_{M_A})$ into itself. By making the constant C in the above expression larger if needed, we obtain similarly that Γ is also a contraction of the ball $B(0, 2\|\phi\|_{M_A})$. This concludes the proof of the existence and uniqueness of the solution $u \in C([0, T_*]; M_A)$. The claim $u = \sum_{k=1}^{\infty} U_k[\phi]$ then follows immediately in the sense of the uniform convergence of partial sums in $C([0, T_*]; M_A)$. \square

The following estimate of the M_A -norm of $U_k[\phi]$ is useful in the proof of the norm inflation phenomenon.

Lemma 5.4. *There exists $C_2 > 0$ (as in Lemma 5.2) such that for any $A \geq 1$, $k \geq 1$, and $\phi \in M_A$, the following holds for all $t > 0$:*

$$\|U_k[\phi](t)\|_{M_A} \leq a_k t^{\frac{k-1}{2}} (C_2 A^{\frac{1}{2}} \|\phi\|_{M_A})^{k-1} \|\phi\|_{M_A}, \quad (5.5)$$

where $\{a_k\}_{k \in \mathbb{N}}$ is the sequence defined by

$$a_1 = 1, \quad a_k = \frac{2}{k-1} \sum_{\substack{k_1, k_2, k_3 \geq 1 \\ k_1 + k_2 + k_3 = k}} a_{k_1} a_{k_2} a_{k_3}, \quad k \geq 2.$$

The proof of Lemma 5.4 is essentially the same as that of an analogous result in [36] (see also [32]), with the only difference that here we are using the unitarity of the operator $e^{-it|D|}$ in M_A , instead of $e^{it\partial_x^2}$. For the sake of completeness, we choose to reproduce this proof here.

Proof. The proof follows by induction. The case $k = 1$ is trivial. Let us now assume that (5.5) holds for $1, 2, \dots, k-1$, and let us prove it for k . By the unitarity of $e^{-it|D|}$ in M_A , Lemma 5.2, and the induction hypothesis, it follows that

$$\begin{aligned} \|U_k[\phi](t)\|_{M_A} &\leq C_2^2 A \sum_{\substack{k_1, k_2, k_3 \geq 1 \\ k_1 + k_2 + k_3 = k}} \int_0^t \|U_{k_1}[\phi]\|_{M_A} \|U_{k_2}[\phi]\|_{M_A} \|U_{k_3}[\phi]\|_{M_A} d\tau \\ &\leq C_2^2 A \frac{2}{k-1} \sum_{\substack{k_1, k_2, k_3 \geq 1 \\ k_1 + k_2 + k_3 = k}} a_{k_1} a_{k_2} a_{k_3} t^{\frac{k-1}{2}} (C_2 A^{\frac{1}{2}} \|\phi\|_{M_A})^{k-3} \|\phi\|_{M_A}^3 \\ &\leq a_k t^{\frac{k-1}{2}} (C_2 A^{\frac{1}{2}} \|\phi\|_{M_A})^{k-1} \|\phi\|_{M_A}. \end{aligned}$$

□

In order to bound sequences $\{a_k\}_{k \in \mathbb{N}}$ with similar properties to the one in Lemma 5.4, we use the following Lemma from [36].

Lemma 5.5 ([36]). *Let $\{a_k\}_{k \in \mathbb{N}}$ be a sequence of nonnegative real numbers for which there exists $C > 0$ such that*

$$a_k \leq C \sum_{\substack{k_1, k_2, k_3 \geq 1 \\ k_1 + k_2 + k_3 = k}} a_{k_1} a_{k_2} a_{k_3}$$

for all $k \geq 2$. Then, the following holds:

$$a_k \leq C_4^{k-1} a_1^k$$

for all $k \geq 1$, where $C_4 = \frac{\pi^2}{6}(9C)^{\frac{1}{2}}$.

The proof of Lemma 5.5 is elementary (by induction) and details can be found in [36]. In particular, by Lemma 5.5, it follows that the sequence $\{a_k\}_{k \in \mathbb{N}}$ from Lemma 5.4 satisfies $a_k \leq C_4^{k-1}$.

In the proof of Theorem 1.3 (iv) we will work with an initial datum ϕ such that

$$\widehat{\phi}(\xi) := R(\mathbf{1}_{N+I_A}(\xi) + \mathbf{1}_{2N+I_A}(\xi)) \text{ for all } \xi \in \mathbb{R}, \quad (5.6)$$

where $N \gg 1$, $1 \ll A \ll N$, $R > 0$ will be chosen later. In other words, $\widehat{\phi}$ is supported on two relatively small intervals centered at high frequencies $N, 2N \gg 1$. It is useful to have the following estimate on the measure of the support of the Fourier transform of $U_k[\phi]$.

Lemma 5.6. *Let $A \geq 1$ and define ϕ as in (5.6). Then, there exists an absolute constant $C > 0$ such that*

$$|\text{supp } \widehat{U_k[\phi]}(t)| \leq C^k A \quad (5.7)$$

for any $k \geq 1$ and $t \geq 0$. In particular, the bound in (5.7) is independent of N .

Proof. The proof is essentially the same as that of an analogous result in [36]. Therefore, we only sketch it here and refer the readers to [36] for details. For k even, (5.7) is trivial since, as noticed earlier, $U_k[\phi] \equiv 0$. For $k = 1$, (5.7) follows easily from the fact that $\widehat{\phi}$ is supported on two intervals of size A centered at N and $2N$ respectively. For $k = 3$, we notice that $\widehat{U_3[\phi]}$ is supported on intervals centered at $q = q_1 - q_2 + q_3$ with $q_1, q_2, q_3 \in \{N, 2N\}$ of size at most $3A$. Therefore,

$$|\text{supp } \widehat{U_3[\phi]}(t)| \leq 2^3 \cdot 3A. \quad (5.8)$$

Arguing by induction, it follows that $\widehat{U_k[\phi]}(t)$ is supported on at most 2^k intervals centered at integers, each of size at most kA and thus, $|\text{supp } \widehat{U_k[\phi]}(t)| \leq 2^k \cdot kA$ for all $k \geq 3$. \square

We are now ready to state and prove the norm inflation property of (1.2) in $H^s(\mathbb{R})$, $s < 0$, that we recall here for convenience.

Proposition 5.7 (Norm inflation property for NHW in $H^s(\mathbb{R})$, $s < 0$). *Let $s < 0$. Then, given $0 < \varepsilon \ll 1$, there exist $\phi \in H^\infty(\mathbb{R})$ with $\|\phi\|_{H^s} < \varepsilon$ and $0 < T < \varepsilon$ such that the solution u of (1.2) with initial condition $u(0) = \phi$ satisfies $\|u(T)\|_{H^s} > \frac{1}{\varepsilon}$.*

As already mentioned above, we choose the initial datum ϕ as in (5.6). Note that $\|\phi\|_{M_A} = CRA^{\frac{1}{2}}$ and $\|\phi\|_{H^s} = C'RA^{\frac{1}{2}}N^s$. The strategy of the proof of Proposition 5.7 is to expand the solution u into the series of $U_k[\phi]$ as in (5.4), and to show that the term $U_3[\phi]$ is much bigger than all the other terms in the series. The conclusion then follows by choosing R , T , and A conveniently in terms of N , such that $\|\phi\|_{H^s} \ll 1$, while $\|U_3[\phi]\|_{H^s} \gg 1$ for a fixed $s < 0$.

The proof of Proposition 5.7 is based on the following two main lemmas. The first lemma gives an upper bound on the H^s -norm of $U_k[\phi]$ for $k \in \mathbb{N}$ and $s \leq 0$.

Lemma 5.8. *Let $s \leq 0$. Then there exists $C > 0$ such that the following hold:*

$$\|U_1[\phi](t)\|_{H^s} \leq CRA^{\frac{1}{2}}N^s \quad (5.9)$$

and

$$\|U_k[\phi](t)\|_{H^s} \leq t^{\frac{k-1}{2}}(CRA)^{k-1}Rg(A), \quad (5.10)$$

for all $k \geq 2$ and all $t \geq 0$, where

$$g(A) := \begin{cases} A^{s+\frac{1}{2}}, & \text{if } -\frac{1}{2} < s \leq 0, \\ (\log A)^{\frac{1}{2}}, & \text{if } s = -\frac{1}{2}, \\ 1, & \text{if } s < -\frac{1}{2}. \end{cases} \quad (5.11)$$

Proof. The first estimate (5.9) is trivial, so we concentrate on (5.10). In what follows, $C > 0$ denotes a generic constant (possibly increasing from line to line). By Hölder's and Young's inequalities, we have that

$$\begin{aligned} \|U_k[\phi](t)\|_{H^s} &\leq \|\langle \xi \rangle^s \mathbf{1}_{\text{supp } \widehat{U_k[\phi]}(t)}\|_{L^2} \|\widehat{U_k[\phi]}(t)\|_{L^\infty} \\ &\leq C \|\langle \xi \rangle^s \mathbf{1}_{\text{supp } \widehat{U_k[\phi]}(t)}\|_{L^2} \sum_{\substack{k_1, k_2, k_3 \geq 1 \\ k_1 + k_2 + k_3 = k}} \int_0^t \|\widehat{U_{k_1}[\phi]}\|_{L^2} \|\widehat{U_{k_2}[\phi]}\|_{L^2} \|\widehat{U_{k_3}[\phi]}\|_{L^2} |\text{supp } \widehat{U_{k_3}[\phi]}(\tau)|^{\frac{1}{2}} d\tau. \end{aligned} \quad (5.12)$$

By (5.7), we first notice that

$$\|\langle \xi \rangle^s \mathbf{1}_{\widehat{\text{supp } U_k[\phi](t)}}\|_{L^2} \lesssim \begin{cases} (C^k A)^{s+\frac{1}{2}}, & \text{if } -\frac{1}{2} < s \leq 0, \\ (\log(C^k A))^{\frac{1}{2}}, & \text{if } s = -\frac{1}{2}, \\ 1, & \text{if } s < -\frac{1}{2}. \end{cases} \quad (5.13)$$

Note that $(\log(C^k A))^{\frac{1}{2}} = (k \log C + \log A)^{\frac{1}{2}} \leq (C')^{k-1} (\log A)^{\frac{1}{2}}$. Secondly, since $k_1, k_2 \geq 1$ and $k_3 \leq k-2$, we have by (5.7) that

$$|\widehat{\text{supp } U_{k_3}[\phi]}(\tau)| \leq C^{k-2} A.$$

Thirdly, by Lemma 5.2, (5.5), and $\|\phi\|_{M_A} \leq C R A^{\frac{1}{2}}$, it follows that

$$\begin{aligned} \|U_{k_j}[\phi](\tau)\|_{L^2} &\leq C \|U_{k_j}[\phi](\tau)\|_{M_A} \leq C a_{k_j} t^{\frac{k_j-1}{2}} (C_2 A^{\frac{1}{2}} \|\phi\|_{M_A})^{k_j-1} \|\phi\|_{M_A} \\ &\leq C a_{k_j} t^{\frac{k_j-1}{2}} (C R A)^{k_j-1} R A^{\frac{1}{2}}. \end{aligned}$$

Combining the last two estimates and using $a_k \leq C_4^{k-1}$ from Lemma 5.5, we obtain that

$$\begin{aligned} &\sum_{\substack{k_1, k_2, k_3 \geq 1 \\ k_1 + k_2 + k_3 = k}} \int_0^t \|\widehat{U_{k_1}[\phi]}\|_{L^2} \|\widehat{U_{k_2}[\phi]}\|_{L^2} \|\widehat{U_{k_3}[\phi]}\|_{L^2} |\widehat{\text{supp } U_{k_3}[\phi]}(\tau)|^{\frac{1}{2}} d\tau \\ &\leq (C^{k-2} A)^{\frac{1}{2}} \frac{2}{k-1} \sum_{\substack{k_1, k_2, k_3 \geq 1 \\ k_1 + k_2 + k_3 = k}} a_{k_1} a_{k_2} a_{k_3} t^{\frac{k-1}{2}} C^3 (C R A)^{k-3} (R A^{\frac{1}{2}})^3 \\ &\leq a_k t^{\frac{k-1}{2}} (C R A)^{k-1} R \leq t^{\frac{k-1}{2}} (C R A)^{k-1} R. \end{aligned} \quad (5.14)$$

The conclusion then follows from (5.12), (5.13), and (5.14). \square

Next, we prove a lower bound for the H^s -norm of $U_3[\phi]$ for $s \leq 0$.

Lemma 5.9. *Let $s \leq 0$. Then, there exists $C > 0$ such that for all $t > 0$, the following holds*

$$\|U_3[\phi](t)\|_{H^s} \geq C t R^3 A^2 g(A), \quad (5.15)$$

where $g(A)$ was defined in (5.11).

Proof. We write

$$\begin{aligned} \widehat{U_3[\phi]}(t)(\xi) &= -i\mu e^{-it|\xi|} \int_0^t e^{i\tau|\xi|} \mathcal{F}\left(|e^{-i\tau|D|}\phi|^2 e^{-i\tau|D|}\phi\right)(\xi) d\tau \\ &= -i\mu e^{-it|\xi|} \iint \left(\int_0^t e^{i\tau(|\xi| - |\xi_1| + |\xi_2| - |\xi_3|)} d\tau \right) \widehat{\phi}(\xi_1) \overline{\widehat{\phi}(\xi_2)} \widehat{\phi}(\xi_3) \mathbf{1}_{\xi = \xi_1 - \xi_2 + \xi_3} d\xi_1 d\xi_3. \end{aligned}$$

From the definition (5.6) of ϕ , we notice that $\widehat{\phi}$ is supported only on positive frequencies. Therefore, the expression under the above integral is supported on $\xi_1, \xi_2, \xi_3 \geq 0$. Next, we restrict our attention to $\xi \in [0, \frac{A}{8})$. In particular, for such ξ , we have

$$|\xi| - |\xi_1| + |\xi_2| - |\xi_3| = \xi - \xi_1 + \xi_2 - \xi_3 = 0.$$

Noticing also that $\xi \in [0, \frac{A}{8}) \subset I_{\frac{A}{4}}$ and $\xi_1, \xi_3 \in N + I_{\frac{A}{4}}$ yield $\xi_2 = \xi_1 + \xi_3 - \xi \in 2N + I_A$ (and thus $\mathbf{1}_{2N+I_A}(\xi_2) \equiv 1$), we then obtain that

$$\begin{aligned}
\left| \widehat{U_3[\phi]}(t)(\xi) \mathbf{1}_{[0, \frac{A}{8})}(\xi) \right| &= t \left| \iint \widehat{\phi}(\xi_1) \overline{\widehat{\phi}(\xi_2)} \widehat{\phi}(\xi_3) \mathbf{1}_{\xi=\xi_1-\xi_2+\xi_3} \mathbf{1}_{[0, \frac{A}{8})}(\xi) d\xi_1 d\xi_3 \right| \\
&= tR^3 \iint \mathbf{1}_{N+I_A}(\xi_1) \mathbf{1}_{2N+I_A}(\xi_2) \mathbf{1}_{N+I_A}(\xi_3) \mathbf{1}_{\xi=\xi_1-\xi_2+\xi_3} \mathbf{1}_{[0, \frac{A}{8})}(\xi) d\xi_1 d\xi_3 \\
&\geq tR^3 \int_{\mathbb{R}} \mathbf{1}_{N+I_{\frac{A}{4}}}(\xi_1) d\xi_1 \int_{\mathbb{R}} \mathbf{1}_{N+I_{\frac{A}{4}}}(\xi_3) d\xi_3 \geq CtR^3 A^2.
\end{aligned}$$

In conclusion, it follows that

$$\begin{aligned}
\|U_3[\phi](t)\|_{H^s} &\geq \|\langle \xi \rangle^s \widehat{U_3[\phi]}(t)(\xi) \mathbf{1}_{[0, \frac{A}{8})}(\xi)\|_{L^2} \geq CtR^3 A^2 \|\langle \xi \rangle^s \mathbf{1}_{[0, \frac{A}{8})}\|_{L^2} \\
&\geq CtR^3 A^2 g(A).
\end{aligned}$$

□

We are now in the position of proving Proposition 5.7.

Proof of Proposition 5.7. We choose ϕ as in (5.6) with the values of R, T, A to be specified later. The condition $\|\phi\|_{H^s} < \varepsilon$ with $0 < \varepsilon \ll 1$ is satisfied if we impose

$$RA^{\frac{1}{2}}N^s \ll 1. \quad (5.16)$$

By Lemma 5.3, there exists a unique solution $u \in C([0, T_*]; M_A)$ of (1.2) admitting the expansion (5.4), where $T_* \sim CA^{-1}\|\phi\|_{M_A}^{-2} = CA^{-2}R^{-2}$. We then require that the time T in Proposition 5.7 satisfies

$$T \leq T_* \sim A^{-2}R^{-2}. \quad (5.17)$$

Under the restrictions (5.16) and (5.17), we now impose that

$$\|U_3[\phi](T)\|_{H^s} \gg \|U_1[\phi](T)\|_{H^s} \quad (5.18)$$

$$\|U_3[\phi](T)\|_{H^s} \gg \sum_{\ell=2}^{\infty} \|U_{2\ell+1}[\phi](T)\|_{H^s}, \quad (5.19)$$

in order to have $\|u(T)\|_{H^s} \gtrsim \|U_3[\phi](T)\|_{H^s}$. The conclusion of the proposition follows if we further impose that

$$\|U_3[\phi](T)\|_{H^s} \gg 1. \quad (5.20)$$

Owing to (5.9) and (5.15), the condition (5.18) amounts to

$$RA^{\frac{1}{2}}N^s \ll TR^3 A^2 g(A) \quad (5.21)$$

and (5.20) amounts to

$$1 \ll TR^3 A^2 g(A), \quad (5.22)$$

where $g(A)$ was defined in (5.11). On the other hand, by (5.10), note that $\sum_{\ell=2}^{\infty} \|U_{2\ell+1}[\phi](T)\|_{H^s}$ behaves like the geometric series

$$\sum_{\ell=2}^{\infty} (CTR^2 A^2)^{\ell} Rg(A).$$

Thus, the series converges and the first term dominates all others provided that $TR^2A^2 \ll 1$. Combining this with (5.15), it follows that (5.19) is satisfied provided that

$$TR^2A^2 \ll 1.$$

To summarize, the conclusion of Proposition 5.7 follows if one can choose $1 \ll A \ll N$, $T \ll 1$, and R such that

$$TR^2A^2 + RA^{\frac{1}{2}}N^s \ll 1 \ll TR^3A^2 \begin{cases} A^{s+\frac{1}{2}}, & \text{if } -\frac{1}{2} < s \leq 0, \\ (\log A)^{\frac{1}{2}}, & \text{if } s = -\frac{1}{2}, \\ 1, & \text{if } s < -\frac{1}{2}. \end{cases} \quad (5.23)$$

For $A \gg 1$ and $s \leq 0$, a stronger condition than (5.23) is:

$$TR^2A^2 + RA^{\frac{1}{2}}N^s \ll 1 \ll TR^3A^2. \quad (5.24)$$

Therefore, Proposition 5.7 also follows if one can choose A, T, R satisfying the stronger condition:

$$T \ll 1, \quad 1 \ll A \ll N, \quad RA^{\frac{1}{2}}N^s \ll 1, \quad \frac{T}{R^2}(RA^{\frac{1}{2}})^4 \ll 1 \ll \frac{T}{R}(RA^{\frac{1}{2}})^4. \quad (5.25)$$

Set now

$$RA^{\frac{1}{2}} = N^\theta, \quad T = N^a, \quad R = N^b, \quad \text{so that} \quad A = N^{2\theta-2b}. \quad (5.26)$$

Then, the conditions (5.25) are satisfied exactly when $\theta < -s$ and

$$\left. \begin{aligned} a &< 0 \\ \max\{0, \theta - \tfrac{1}{2}\} &< b < \theta \\ a - 2b + 4\theta &< 0 \\ a - b + 4\theta &> 0 \end{aligned} \right\} \quad (5.27)$$

In particular, notice that this imposes $0 < \theta < -s$, which is possible provided that $s < 0$. Choosing, for example, $0 < \theta < \min(-s, \frac{1}{2})$ it follows that the solution of (5.27) is nonempty. Namely, it is the interior of the triangle ABC (see Figure 2 below) whose vertices have (a, b) -coordinates given by

$$A: (-4\theta, 0), \quad B: (-3\theta, \theta), \quad C: (-2\theta, \theta). \quad (5.28)$$

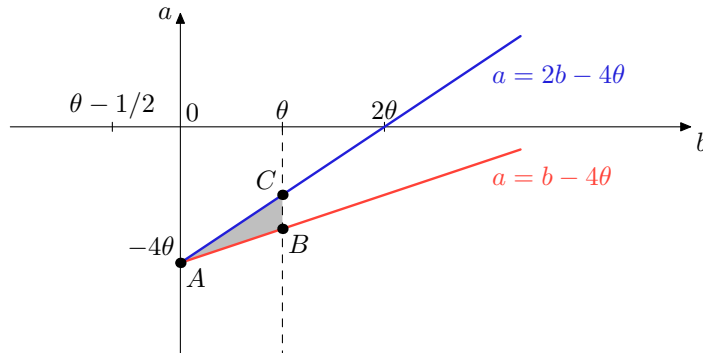


FIGURE 2. Triangle ABC .

This shows that for $s < 0$ it is indeed possible to choose T, R , and A as in (5.26), satisfying (5.25). Therefore, (1.2) has the norm inflation property in $H^s(\mathbb{R})$, $s < 0$, which concludes the proof of Proposition 5.7. \square

Remark 5.10. (i). Due to the conservation of the L^2 -norm by the flow of (1.2), Proposition 5.7 does not hold for $s = 0$.

(ii). The cubic Szegő equation also has the norm inflation property in $H_+^s(\mathbb{R})$, $s < 0$. The proof is essentially the same as that of Proposition 5.7, with the exception that the operator $e^{-it|D|}$ is replaced by the identity operator.

6. NORM INFLATION FOR FRACTIONAL NLS IN $H^s(\mathbb{R})$, $s < 0$

We end this paper with the proof of Theorem 1.5. Namely, we show that the norm inflation phenomenon in $H^s(\mathbb{R})$, $s < 0$, also occurs for other cubic fractional nonlinear Schrödinger equations.

Proof of Theorem 1.5. The proof follows the same lines as that of Proposition 5.7, with the operator $e^{-it|D|}$ replaced by $e^{-it|D|^\beta}$. In particular, we consider the same initial condition $u(0) = \phi$ as in (5.6), the local existence time $T_* \sim A^{-2}R^{-2}$ is the same as for (1.2), and the solution of (1.1) is again given by an expansion $u = \sum U_k[\phi]$. Evidently, the condition $\|\phi\|_{H^s} \ll 1$ remains $RA^{\frac{1}{2}}N^s \ll 1$. The only essential difference appears in the proof of Lemma 5.9. More precisely, the time integral in the proof of Lemma 5.9 becomes

$$\int_0^t e^{i\tau(|\xi|^\beta - |\xi_1|^\beta + |\xi_2|^\beta - |\xi_3|^\beta)} d\tau$$

with $\xi - \xi_1 + \xi_2 - \xi_3 = 0$ and $\xi_1, \xi_2, \xi_3 > 0$. For $\beta \neq 1$, one can no longer choose ξ so that the above phase vanishes. Instead, we notice that for $\xi \in I_{\frac{A}{4}}$, $\xi_1, \xi_3 \in N + I_A$, and $\xi_2 \in 2N + I_A$, one has $|\xi|^\beta - |\xi_1|^\beta + |\xi_2|^\beta - |\xi_3|^\beta = O(N^\beta)$. Therefore, if we choose $|t| \ll N^{-\beta}$, we obtain

$$\left| \int_0^t e^{i\tau(|\xi|^\beta - |\xi_1|^\beta + |\xi_2|^\beta - |\xi_3|^\beta)} d\tau \right| \gtrsim t,$$

which suffices for the purposes of Lemma 5.9. The extra condition $T \ll N^{-\beta}$, however, needs to be added to the previous requirements $1 \ll A \ll N$ and (5.23) from the proof of Proposition 5.7.

Case I: $s \leq -\frac{1}{2}$.

As in the proof of Proposition 5.7, we verify the stronger conditions (5.25) instead of (5.23), to which we add the requirement $T \ll N^{-\beta}$. Setting

$$RA^{\frac{1}{2}} = N^\theta, \quad T = N^a, \quad R = N^b, \quad \text{so that} \quad A = N^{2\theta-2b},$$

as in the proof of Proposition 5.7, these are satisfied exactly when $\theta < -s$ and

$$\left. \begin{aligned} a &< -\beta \\ \max\{0, \theta - \tfrac{1}{2}\} &< b < \theta \\ a - 2b + 4\theta &< 0 \\ a - b + 4\theta &> 0 \end{aligned} \right\} \quad (6.1)$$

Subcase I.a: $s \leq -\frac{1}{2}$ and $\theta \geq \frac{1}{2}$.

In this subcase, the solution of (6.1) is the intersection of the half plane $a < -\beta$ with the interior of the quadrilateral $BCDE$ in Figure 3, whose vertices have (a, b) -coordinates

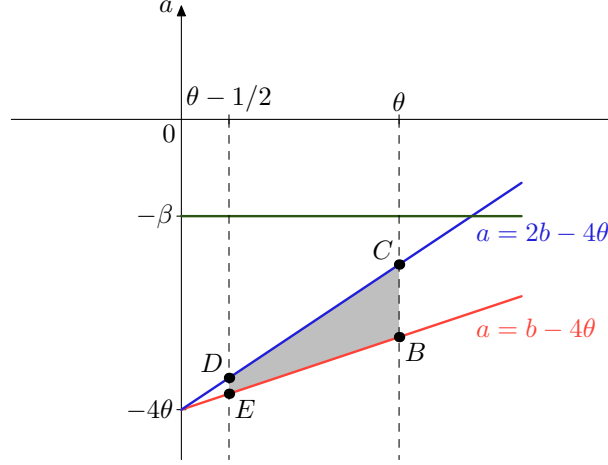


FIGURE 3. Quadrilateral $BCDE$.

given by

$$B: (-3\theta, \theta), \quad C: (-2\theta, \theta), \quad D: (-2\theta - 1, \theta - \frac{1}{2}), \quad E: (-3\theta - \frac{1}{2}, \theta - \frac{1}{2}).$$

This intersection is nonempty if the vertex E of the quadrilateral $BCDE$ lies in the half plane $a < -\beta$, that is if $-\beta > -3\theta - \frac{1}{2}$. This imposes the following constraints on θ : $\frac{2\beta-1}{6} < \theta < -s$ and $\theta \geq \frac{1}{2}$. Such θ exists provided that either $0 < \beta \leq 2$ and $s < -\frac{1}{2}$, or $\beta > 2$ and $s < \frac{1-2\beta}{6}$.

In conclusion, for $0 < \beta \leq 2$ and $s < -\frac{1}{2}$, or $\beta > 2$ and $s < \frac{1-2\beta}{6}$, conditions (5.25) and $T \ll N^{-\beta}$ are indeed satisfied.

Subcase I.b: $s \leq -\frac{1}{2}$ and $0 < \theta < \frac{1}{2}$.

The solution of (6.1), in this subcase, is the intersection of the half plane $a < -\beta$ with the interior of the triangle ABC (see Figure 2 above) whose vertices have (a, b) -coordinates given by

$$A: (-4\theta, 0), \quad B: (-3\theta, \theta), \quad C: (-2\theta, \theta).$$

This intersection is nonempty if the vertex A of the triangle lies in the half plane $a < -\beta$, that is $-\beta > -4\theta$. This dictates the choice $\frac{\beta}{4} < \theta < \frac{1}{2}$, which is possible only if $\beta < 2$.

Therefore, we obtain that if $0 < \beta < 2$ and $s \leq -\frac{1}{2}$ conditions (5.25) and $T \ll N^{-\beta}$ are satisfied.

Case II: $s = -\frac{1}{2}$ and $\beta = 2$.

In this case, instead of verifying the stronger conditions (5.25), we verify conditions (5.23). We choose, as in [36]:

$$T = \frac{1}{N^2(\log N)^{\frac{1}{6}}}, \quad R = 1, \quad A = \frac{N}{(\log N)^{\frac{1}{12}}}.$$

Then, $RA^{\frac{1}{2}}N^s = \frac{1}{(\log N)^{\frac{1}{24}}} \ll 1$, $TR^2A^2 = \frac{1}{(\log N)^{\frac{1}{3}}} \ll 1$, and $TR^3A^2 \cdot (\log A)^{\frac{1}{2}} \sim (\log N)^{\frac{1}{6}} \gg 1$.

1. This shows that $T \ll N^{-2}$, $1 \ll A \ll N$, and (5.23) are indeed satisfied.

Case III: $-\frac{1}{2} < s < 0$.

From (5.23), it follows that in this case it is enough to verify the following conditions:

$$T \ll N^{-\beta}, \quad 1 \ll A \ll N, \quad TR^2A^2 + RA^{\frac{1}{2}}N^s \ll 1 \ll TR^3A^{\frac{5}{2}+s}. \quad (6.2)$$

With the choice of T , R , and A from (5.26), it follows that these conditions are satisfied exactly when $\theta < -s$ and

$$\left. \begin{aligned} a &< -\beta \\ \theta - \frac{1}{2} < b &< \theta \\ a - 2b + 4\theta &< 0 \\ a - (2 + 2s)b + (5 + 2s)\theta &> 0 \end{aligned} \right\} \quad (6.3)$$

Notice that $\theta < -s$ and $s > -\frac{1}{2}$ yield $\theta < \frac{1}{2}$.

Secondly, we notice that the above conditions cannot be simultaneously satisfied if $\theta \leq 0$. Indeed, assume that $\theta \leq 0$. The last two conditions in (6.3) require that:

$$(2 + 2s)b - (5 + 2s)\theta < a < 2b - 4\theta,$$

and thus $b > \frac{1+2s}{2s}\theta \geq 0$. On the other hand, the second condition in (6.3) would impose that $b \in (\theta - \frac{1}{2}, \theta) \subset (-\infty, 0)$. This is a contradiction, and therefore (6.3) does not have a solution if $\theta \leq 0$.

From now on we assume that $0 < \theta < \frac{1}{2}$. Then, the solution of (6.3) is the intersection of the half plane $a < -\beta$ with the interior of the triangle $A'BC$ in Figure 4 below,

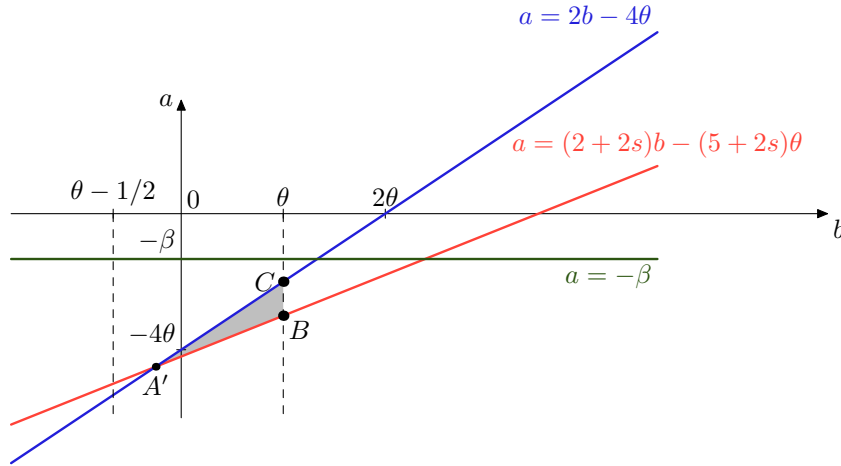


FIGURE 4. Triangle $A'BC$.

whose vertices have (a, b) -coordinates given by

$$A': \left(\frac{1-2s}{s}\theta, \frac{1+2s}{2s}\theta \right), \quad B: (-3\theta, \theta), \quad C: (-2\theta, \theta).$$

This intersection is nonempty if the vertex A' of the triangle lies in the half plane $a < -\beta$, that is if $-\beta > \frac{1-2s}{s}\theta$. Combining this with $\theta < -s$ and $-\frac{1}{2} < s < 0$ yields $-\frac{1}{2} < s < \min(\frac{1-\beta}{2}, 0)$ and $\beta < 2$.

Therefore, if $0 < \beta < 2$ and $-\frac{1}{2} < s < \min(\frac{1-\beta}{2}, 0)$, the conditions (6.2) are satisfied.

Collecting the information we obtained in Cases I, II, and III above, we conclude that the norm inflation phenomenon occurs for (1.1) in the following cases: $0 < \beta < 1$ and $s < 0$; $1 \leq \beta < 2$ and $s < s_{\text{crit}}$; $\beta = 2$ and $s \leq s_{\text{crit}}$; $\beta > 2$ and $s < \frac{1-2\beta}{6}$. \square

Remark 6.1. The method developed in this section cannot be used to decide on the norm inflation property of (1.1) in $H^s(\mathbb{R})$ in the following two cases: $1 < \beta < 2$ and $s = s_{\text{crit}}$; $\beta > 2$ and $s = \frac{1-2\beta}{6}$.

Proof. We follow an argument from [45, Remark B.5]. In view of the requirements $A \ll N$ and $T \ll N^{-\beta}$ of our method, we write $A = EN$ with $E \ll 1$ and $T = FN^{-\beta}$ with $F \ll 1$.

For $1 < \beta < 2$, notice that $-\frac{1}{2} < s_{\text{crit}} = \frac{1-\beta}{2} < 0$. Set $G := RA^{\frac{1}{2}}N^{s_{\text{crit}}} \ll 1$. Then,

$$TR^3A^{\frac{5}{2}+s_{\text{crit}}} = FN^{-\beta}(RA^{\frac{1}{2}}N^{s_{\text{crit}}})^3N^{-3s_{\text{crit}}}A^{1+s_{\text{crit}}} = E^{1+s_{\text{crit}}}FG^3 \ll 1.$$

In particular, there is no choice of T , R , and A such that $RA^{\frac{1}{2}}N^{s_{\text{crit}}} \ll 1$ and $TR^3A^{\frac{5}{2}+s} \gg 1$, as required by conditions (5.23) for the norm inflation property.

For $\beta > 2$, notice that $\frac{1-2\beta}{6} < -\frac{1}{2}$. Set $H := TR^3A^2 \gg 1$. This yields $R = E^{-\frac{2}{3}}F^{-\frac{1}{3}}H^{\frac{1}{3}}N^{\frac{\beta-2}{3}}$, and thus

$$RA^{\frac{1}{2}}N^{\frac{1-2\beta}{6}} = E^{-\frac{1}{6}}F^{-\frac{1}{3}}H^{\frac{1}{3}} \gg 1.$$

In particular, there is no choice of T , R , and A such that $TR^3A^2 \gg 1$ and $RA^{\frac{1}{2}}N^{\frac{1-2\beta}{6}} \ll 1$, as required by conditions (5.23) for the norm inflation property. \square

Acknowledgement. A.C. was partially supported by a Whittaker Research Fellowship at the University of Edinburgh and by the European Research Council (grant agreement no. 616797). O.P. was partially supported by the National Science Foundation under grant no. DMS-1440140 while she was in residence at the Mathematical Sciences Research Institute in Berkeley, California, during the Fall 2015 semester. The authors would like to thank Tadahiro Oh, Nobu Kishimoto, Nikolay Tzvetkov, and Vladimir Georgiev for their generous help.

REFERENCES

- [1] T. Alazard, R. Carles, *Loss of regularity for supercritical nonlinear Schrödinger equations*, Math. Ann. 343 (2009), no. 2, 397–420.
- [2] I. Bejenaru, T. Tao, *Sharp well-posedness and ill-posedness results for a quadratic non-linear Schrödinger equation*, J. Funct. Anal. 233 (2006), no. 1, 228–259.
- [3] Á. Bényi, K. Okoudjou, *Local well-posedness of nonlinear dispersive equations on modulation spaces*, Bull. Lond. Math. Soc. 41 (2009), no. 3, 549–558.
- [4] H.A. Biagioni, F. Linares, *Ill-posedness for the derivative Schrödinger and generalized Benjamin-Ono equations*, Trans. Amer. Math. Soc. 353 (2001), no. 9, 3649–3659.

- [5] B. Birnir, C. Kenig, G. Ponce, N. Svanstedt, L. Vega, *On the ill-posedness of the IVP for the generalized Korteweg-de Vries and nonlinear Schrödinger equations*, J. London Math. Soc. (2) 53 (1996), no. 3, 551–559.
- [6] J. Bourgain, *Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations I. Schrödinger equations*, Geom. Funct. Anal. 3 (1993), no. 2, 107–156.
- [7] J. Bourgain, *Periodic Korteweg de Vries equation with measures as initial data*, Selecta Math. (N.S.) 3 (1997), no. 2, 115–159.
- [8] N. Burq, P. Gérard, N. Tzvetkov, *An instability property of the nonlinear Schrödinger equation on \mathbb{S}^d* , Math. Res. Lett. 9 (2002), no. 2-3, 323–335.
- [9] N. Burq, P. Gérard, N. Tzvetkov, *Multilinear eigenfunction estimates and global existence for the three dimensional nonlinear Schrödinger equations*, Ann. Sci. École Norm. Sup. (4) 38 (2005), no. 2, 255–301.
- [10] N. Burq, N. Tzvetkov, *Random data Cauchy theory for supercritical wave equations. I. Local theory*, Invent. Math. 173 (2008), no. 3, 449–475.
- [11] R. Carles, E. Dumas, C. Sparber, *Multiphase weakly nonlinear geometric optics for Schrödinger equations*, SIAM J. Math. Anal. 42 (2010), no. 1, 489–518.
- [12] R. Carles, E. Dumas, C. Sparber, *Geometric optics and instability for NLS and Davey-Stewartson models*, J. Eur. Math. Soc. (JEMS) 14 (2012), no. 6, 1885–1921.
- [13] R. Carles, T. Kappeler, *Norm-inflation for periodic NLS equations in negative Sobolev spaces*, arXiv:1507.04218.
- [14] Y. Cho, G. Hwang, S. Kwon, S. Lee, *Well-posedness and ill-posedness for the cubic fractional Schrödinger equations*, Discrete Contin. Dyn. Syst. 35 (2015), no. 7, 2863–2880.
- [15] M. Christ, J. Colliander, T. Tao, *Asymptotics, frequency modulation, and low regularity ill-posedness for canonical defocusing equations*, Amer. J. Math. 125 (2003), no. 6, 1235–1293.
- [16] M. Christ, J. Colliander, T. Tao, *Ill-posedness for nonlinear Schrödinger and wave equations*, arXiv:math/0311048.
- [17] M. Christ, J. Colliander, T. Tao, *Instability of the periodic nonlinear Schrödinger equation*, arXiv:math/0311227v1.
- [18] S. Demirbas, M. B. Erdoğan, N. Tzirakis, *Existence and uniqueness theory for the fractional Schrödinger equation on the torus*, arXiv:1312.5249.
- [19] A. Elgart, B. Schlein, *Mean field dynamics of boson stars*, Commun. Pure Appl. Math. 60 (2007), no. 4, 500–545.
- [20] H. Feichtinger, *Modulation spaces of locally compact Abelian groups*, Technical report, University of Vienna (1983). in Proc. Internat. Conf. on Wavelets and Applications (Chennai, 2002), R. Radha, M. Krishna, S. Thangavelu (eds.), New Delhi Allied Publishers (2003), 156.
- [21] H. Feichtinger, K. Gröchenig, *Banach spaces related to integrable group representations and their atomic decompositions, I*, J. Func. Anal. 86 (1989), 307–340.
- [22] H. Feichtinger and K. Gröchenig, *Banach spaces related to integrable group representations and their atomic decompositions, II*, Monatsh. Math. 108 (1989), 129–148.
- [23] J. Fröhlich, E. Lenzmann, *Blowup for nonlinear wave equations describing boson stars*, Commun. Pure Appl. Math. 60 (2007), no. 11, 1691–1705.
- [24] V. Georgiev, N. Tzvetkov, N. Visciglia, *On the regularity of the flow map associated with the 1D cubic periodic half-wave equation*, Diff. Integral Equations 29 (2016), no. 1/2, 183–200.
- [25] P. Gérard, S. Grellier, *The cubic Szegő equation*, Ann. Sci. Éc. Norm. Supr. (4) 43 (2010), no. 5, 761–810.
- [26] P. Gérard, S. Grellier, *Effective integrable dynamics for a certain nonlinear wave equation*, Anal. PDE 5 (2012), no. 5, 1139–1155.
- [27] P. Gérard, E. Lenzmann, O. Pocovnicu, P. Raphaël, *A two soliton with transient turbulent regime for the focusing cubic half-wave equation on \mathbb{R}* , preprint.
- [28] Z. Guo, T. Oh, *Non-existence of solutions for the periodic cubic NLS below L^2* , arXiv:1510.06208.
- [29] S. Herr, E. Lenzmann, *The Boson star equation with initial data of low regularity*, Nonlinear Anal. 97 (2014), 125–137.
- [30] S. Ibrahim, M. Majdoub, N. Masmoudi, *Ill-posedness of H^1 -supercritical waves*, C. R. Math. Acad. Sci. Paris 345 (2007), no. 3, 133–138.
- [31] A. Ionescu, F. Pusateri, *Nonlinear fractional Schrödinger equations in one dimension*, J. Funct. Anal. 266 (2014), no. 1, 139–176.

- [32] T. Iwabuchi, T. Ogawa, *Ill-posedness for the nonlinear Schrödinger equation with quadratic nonlinearity in low dimensions*, Trans. Amer. Math. Soc. 367 (2015), no. 4, 2613–2630.
- [33] T. Iwabuchi, K. Uriya, *Ill-posedness for the quadratic nonlinear Schrödinger equation with nonlinearity $|u|^2$* , Commun. Pure Appl. Anal. 14 (2015), no. 4, 1395–1405.
- [34] C. Kenig, G. Ponce, L. Vega, *On the ill-posedness of some canonical dispersive equations*, Duke Math. J. 106 (2001), no. 3, 617–633.
- [35] K. Kirkpatrick, E. Lenzmann, G. Staffilani, *On the continuum limit for discrete NLS with long-range interactions*, Commun. Math. Phys. 317 (2013), no. 3, 563–591.
- [36] N. Kishimoto, *A remark on norm inflation for nonlinear Schrödinger equations*, preprint.
- [37] H. Koch, N. Tzvetkov, *Nonlinear wave interactions for the Benjamin-Ono equation*, Int. Math. Res. Not. 2005, no. 30, 1833–1847.
- [38] J. Krieger, E. Lenzmann, P. Raphaël, *Nondispersive solutions to the L^2 -critical half-wave equation*, Arch. Ration. Mech. Anal. 209 (2013), no. 1, 61–129.
- [39] G. Lebeau, *Optique non linéaire et ondes sur critiques*, (French) [Nonlinear optics and supercritical waves] Séminaire: Équations aux Dérivées Partielles, 1999–2000, Exp. No. IV, 13 pp., Sémin. Équ. Dériv. Partielles, École Polytech., Palaiseau, 2000.
- [40] G. Lebeau, *Perte de régularité pour les équations d’ondes sur-critiques*, (French) [Loss of regularity for super-critical wave equations] Bull. Soc. Math. France 133 (2005), no. 1, 145–157.
- [41] H. Lindblad, *A sharp counterexample to the local existence of low-regularity solutions to nonlinear wave equations*, Duke Math. J. 72 (1993), no. 2, 503–539.
- [42] A. Majda, D. McLaughlin, E. Tabak, *A one-dimensional model for dispersive wave turbulence*, J. Nonlinear Sci. 7 (1997), no. 1, 9–44.
- [43] L. Molinet, *On ill-posedness for the one-dimensional periodic cubic Schrödinger equation*, Math. Res. Lett. 16 (2009), no. 1, 111–120.
- [44] T. Oh, C. Sulem, *On the one-dimensional cubic nonlinear Schrödinger equation below L^2* , Kyoto J. Math. 52 (2012), no. 1, 99–115.
- [45] T. Oh, Y. Wang, *On the ill-posedness of the cubic nonlinear Schrödinger equation on the circle*, arXiv:1508.00827.
- [46] T. Oh, *A remark on norm inflation with general initial data for the cubic nonlinear Schrödinger equations in negative Sobolev spaces*, to appear in Funkcial. Ekvac.
- [47] O. Pocovnicu, *Explicit formula for the solution of the Szegő equation on the real line and applications*, Discrete Contin. Dyn. Syst. 31 (2011), no. 3, 607–649.
- [48] O. Pocovnicu, *Traveling waves for the cubic Szegő equation on the real line*, Anal. PDE, 4 (2011), no. 3, 379–404.
- [49] O. Pocovnicu, *First and second order approximations for a nonlinear wave equation*, J. Dynam. Differential Equations, article no. 9286 (2013), 29 pp, DOI 10.1007/s10884-013-9286-5.
- [50] N. Tzvetkov, *Remark on the local ill-posedness for KdV equation*, C. R. Acad. Sci. Paris Sr. I Math. 329 (1999), no. 12, 1043–1047.
- [51] N. Tzvetkov, *Ill-posedness issues for nonlinear dispersive equations. Lectures on nonlinear dispersive equations*, 63–103, GAKUTO Internat. Ser. Math. Sci. Appl., 27, Gakktōsho, Tokyo, 2006.
- [52] B. Xia, *Generic ill-posedness for wave equation of power type on 3D torus*, arXiv:1507.07179.

ANTOINE CHOFFRUT, SCHOOL OF MATHEMATICS, THE UNIVERSITY OF EDINBURGH, JAMES CLERK MAXWELL BUILDING, THE KING’S BUILDINGS, PETER GUTHRIE TAIT ROAD, EDINBURGH, EH9 3FD, SCOTLAND AND MATHEMATICS INSTITUTE, UNIVERSITY OF WARWICK, ZEEMAN BUILDING, COVENTRY CV4 7AL, UNITED KINGDOM

E-mail address: A.Choffrut@warwick.ac.uk

OANA POCOVNICU, DEPARTMENT OF MATHEMATICS, PRINCETON UNIVERSITY, FINE HALL, WASHINGTON RD., PRINCETON, NJ 08544-1000, USA, AND DEPARTMENT OF MATHEMATICS, HERIOT-WATT UNIVERSITY AND THE MAXWELL INSTITUTE FOR THE MATHEMATICAL SCIENCES, EDINBURGH EH14 4AS, UNITED KINGDOM

E-mail address: o.pocovnicu@hw.ac.uk